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Inverse problem for the gravimetric modeling of the crust-mantle density contrast

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Abstract: The gravimetric inverse problem for finding the Moho density contrast is formulated in this study. The solution requires that the crust density structure and the Moho depths are a priori known, for instance, from results of seismic studies. The relation between the isostatic gravity data (i.e., the complete-crust stripped isostatic gravity disturbances) and the Moho density contrast is defined by means of the Fredholm integral equation of the first kind. The closed analytical solution of the integral equation is given. Alternative expressions for solving the inverse problem of isostasy are defined in frequency domain. The isostatic gravity data are computed utilizing methods for a spherical harmonic analysis and synthesis of the gravity field. For this purpose, we define various spherical functions, which define the crust density structures and the Moho interface globally.

Key words: crust, density contrast, gravity, isostasy, Moho interface

1. Introduction

In gravimetric studies of the isostasy, two basic concepts have been commonly adopted, assuming that the topographic mass surplus and the oceanic mass deficiency are compensated either by a variable thickness or density of compensation. In the Pratt–Hayford model, the isostatic mass balance is attained by a variable density of compensation (*Pratt, 1855; Hayford, 1909; Hayford and Bowie, 1912*). The Airy–Heiskanen model assumes that a depth of compensation is variable (*Airy, 1855; Heiskanen and Vening Meinesz, 1958*). *Vening Meinesz (1931)* modified the Airy–Heiskanen theory by introducing a regional instead of local compensation. *Moritz (1990)* generalized Vening Meinesz's inverse problem for a global isostatic compensation mechanism and applied a spherical approximation to the problem. *Sjöberg (2009)* formulated Moritz's problem, called herein the Vening

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Meinesz-Moritz (VMM) problem of isostasy, as that of solving a non-linear Fredholm integral equation of the first kind. The seismic studies revealed that not only the Moho depth but also the Moho density contrast varies significantly (cf. *Geiss, 1987; Martinec, 1994; Kaban et al., 2003; Sjöberg and Bagherbandi, 2011*). The isostatic model should then be formulated based on assumption that both these quantities (i.e., Moho depth and density contrast) are variable. Following this concept, *Sjöberg and Bagherbandi (2011)* generalized the VMM problem. They developed and applied a least-squares approach, which combined seismic and gravity data in the isostatic inverse scheme for a simultaneous estimation of the Moho depth and density contrast. Later, they also presented and applied the non-isostatic correction to model for discrepancies between the isostatic and seismic models (cf. *Bagherbandi and Sjöberg, 2012*).

In gravimetric studies, the anomalous density structure not only within the crust but essentially within the whole lithosphere should be modeled (cf. e.g., *Kaban et al., 1999; Tenzer et al., 2009a, 2012c*). Moreover, large portion of the isostatic mass balance is attributed to variable sub-lithosphere mantle density structure, which has significant effect especially on a long-wavelength part of the isostatic gravity spectra and consequently on the respective Moho geometry (cf. *Sjöberg, 2009*). The gravitational field generated by all known anomalous density structures should thus be modeled and subsequently removed from observed gravity field prior to solving the gravimetric inverse problem.

In this study, the gravimetric inverse problem for finding the Moho density contrast is formulated using available models of the crust density structure and the Moho geometry. The solution is numerically realized in two steps. First, the gravimetric forward modeling is applied to compute the isostatic gravity data. These isostatic gravity data are then used to find the Moho density contrast based on solving the inverse problem of isostasy. In the absence of a reliable global mantle density model, the gravimetric problem is here formulated only for the crust mass balance.

2. Functional model

We formulate the functional model for finding the Moho density contrast (by means of Newton's volumetric integral) in the following form

$$\delta g^{cs}(r,\Omega) = -\mathrm{G} \iint\limits_{\Omega'\in\Phi} \Delta\rho^{\mathrm{c/m}}(\Omega') \int_{r'=\mathrm{R}-D(\Omega')}^{\mathrm{R}} \frac{\partial\, \ell^{-1}(r,\psi,\, r')}{\partial\, r}\, r'^2\, \mathrm{d}r'\mathrm{d}\Omega', \quad (1)$$

where $\mathrm{G} = 6.674 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$ is Newton's gravitational constant; $\mathrm{R} = 6371 \times 10^3$ m is the Earth's mean radius (which approximates the geocentric radii of the geoid surface); ℓ is the Euclidean spatial distance of two points (r,Ω) and (r',Ω'), ψ is the respective spherical distance; and $\mathrm{d}\Omega' = \cos\phi'\mathrm{d}\phi'\mathrm{d}\lambda'$ is the infinitesimal surface element on the unit sphere. The 3-D position is defined in the system of spherical coordinates (r,Ω); where r is the spherical radius, and $\Omega = (\phi,\lambda)$ denotes the spherical direction with the spherical latitude ϕ and longitude λ. The full spatial angle is denoted as $\Phi = \{\Omega' = (\phi',\lambda') : \phi' \in [-\pi/2,\, \pi/2] \wedge \lambda' \in [0,\, 2\pi)\}$. The geocentric radius r of the observation surface point is computed as $r \cong \mathrm{R} + H$, where H is the topographic height. The Moho depths D' are taken relative to the sphere of radius R.

The unknown parameter in Eq. (1) is the laterally varying Moho density contrast $\Delta\rho^{\mathrm{c/m}}$, while it is assumed that the Moho depths D' are a priori known (for instance, from results of seismic surveys). The consolidated crust-stripped gravity disturbances δg^{cs} on the left-hand side of Eq. (1) are obtained from the gravity disturbances δg after applying the topographic and stripping gravity corrections of major known anomalous crust density structures. The global results of the topographic and crust components stripping gravity corrections and the step-wise consolidated crust-stripped gravity data were presented and discussed in *Tenzer et al. (2009b, 2012c). Tenzer et al. (2011)* demonstrated that these gravity data have a correlation with the Moho geometry of 0.96; see also *Tenzer et al. (2009b)*.

The volumetric integral on the right-hand side of Eq. (1) is further divided into two constituents, which are defined for the average (constant) value of the Moho density contrast $\Delta\rho^{\mathrm{c/m}}$ and the correction term $\delta\Delta\rho^{\mathrm{c/m}}(\Omega') = \Delta\rho^{\mathrm{c/m}}(\Omega') - \Delta\rho^{\mathrm{c/m}}$. Hence,

$$\begin{aligned}\delta g^{cs}(r,\Omega) = &- \mathrm{G}\, \Delta\rho^{\mathrm{c/m}} \iint\limits_{\Omega'\in\Phi} \int_{r'=\mathrm{R}-D(\Omega')}^{\mathrm{R}} \frac{\partial\, \ell^{-1}(r,\psi,\, r')}{\partial\, r}\, r'^2\, \mathrm{d}r'\mathrm{d}\Omega' - \\ &- \mathrm{G} \iint\limits_{\Omega'\in\Phi} \delta\Delta\rho^{\mathrm{c/m}}(\Omega') \int_{r'=\mathrm{R}-D(\Omega')}^{\mathrm{R}} \frac{\partial\, \ell^{-1}(r,\psi,\, r')}{\partial\, r}\, r'^2\, \mathrm{d}r'\mathrm{d}\Omega'. \quad (1a)\end{aligned}$$

The first constituent on the right-hand side of Eq. (1a) is the complete-crust stripped gravitational correction g^i. Subtracting this term from the consolidated crust-stripped gravity disturbances δg^{cs}, we arrive at

$$\delta g^m(r,\Omega) = -\,\mathrm{G}\iint_{\Omega'\in\Phi} \delta\Delta\rho^{\mathrm{c/m}}(\Omega')\int_{r'=\mathrm{R}-D(\Omega')}^{\mathrm{R}} \frac{\partial\,\ell^{-1}(r,\psi,\,r')}{\partial\,r} r'^{\,2}\,\mathrm{d}r'\mathrm{d}\Omega', \quad (1b)$$

where $\delta g^m = \delta g^{cs} - g^i$ is the complete crust-stripped (relative to the upper mantle density) isostatic gravity disturbance. These isostatic gravity disturbances describe the gravity field generated by the regularized Earth of which topography is removed and the actual crust structures (beneath the geoid surface down to the Moho interface) are replaced by a homogeneous crust model of the adopted (constant) reference density of the upper(most) mantle (see *Tenzer et al., 2009a*). For the reference crust density of 2670 kg/m^3, the average value of the Moho density contrast $\Delta\rho^{\mathrm{c/m}}$ of 485 kg/m^3 can be recommended. This value was estimated based on minimizing the correlation between the gravity and Moho depth data (cf. *Tenzer et al., 2011*). It closely agrees with the value of 480 kg/m^3 adopted in the definition of the Preliminary Reference Earth Model (*Dziewonski and Anderson, 1981, Table 1*), which was derived based on results of seismic studies. The isostatic gravity disturbances δg^m on the right-hand side of Eq. (1b) are used as the input gravity data. The unknown parameters to be estimated are the correction terms $\delta\Delta\rho^{\mathrm{c/m}}$. The expressions for the gravimetric forward modeling of δg^{cs} and δg^m are reviewed in Appendices A and B, respectively. In Sections 3 and 4, we derive the solution to the gravimetric inverse problem (Eq. 1b) in spatial and spectral domains.

The definition of the complete crust-stripped isostatic gravity disturbances δg^m has analogy with the definition of the isostatic gravity data in the VMM model (see *Sjöberg, 2009*). The principal difference between these two definitions is in using the gravity disturbances instead of gravity anomalies. Moreover, the definition of δg^m is based on minimizing their correlation with the Moho geometry. *Tenzer and Bagherbandi (2012e)* reformulated the VMM inverse problem of isostasy for the isostatic gravity disturbances. They also demonstrated that the results obtained using the isostatic gravity disturbances have a better agreement with the seismic Moho model. *Sjöberg (2013)* summarized the definitions of the isostatic gravity field quantities for the potential and gravity data types. He also gave a theoretical explanation to the numerical results of *Tenzer and Bagherbandi (2012e)*.

3. Spatial form

Let us first define the radial integral kernel function K as

$$K\left(r,\psi,r'\right) = -\int_{r'=\mathrm{R}-D(\Omega')}^{\mathrm{R}} \frac{\partial\, \ell^{-1}\left(r,\psi,\, r'\right)}{\partial\, r}\, r'^{2}\,\mathrm{d}r'. \tag{2}$$

Substitution from Eq. (2) back to Eq. (1b) then yields

$$\delta g^{m}\left(r,\Omega\right) \cong \mathrm{G}\iint\limits_{\Omega'\in\Phi} \delta\Delta\rho^{\mathrm{c/m}}\left(\Omega'\right)\; K\left(r,\psi,r'\right)\;\mathrm{d}\Omega'. \tag{3}$$

The expression in Eq. (3) is a Fredholm integral equation of the first kind. The closed analytical form of K reads (*Martinec, 1998, Eq. 3.54*)

$$K\left(r,\psi,r'\right) = \Bigg|\;\left[\left(r'^{2}+3r^{2}\right)\; t + \left(1-6t^{2}\right)\, r\, r'\right]\ell^{-1}\left(r,\psi,\, r'\right) + \\ + r\;\left(3t-1\right)\;\ln\left|\, r' - rt + \ell\left(r,\psi,\, r'\right)\,\right| \Bigg|_{r'=\mathrm{R}}^{\mathrm{R}-D'}, \tag{4}$$

where $D(\Omega') \equiv D'$; $t = \cos\psi$, and $\cos\psi = \sin\phi\sin\phi' + \cos\phi\cos\phi'\cos(\lambda'-\lambda)$.

Substitution of the integral limits in Eq. (4) yields

$$K\left(r,\psi,r'\right) = \left[\left(\mathrm{R}^{2} - 2\mathrm{R}D' + D'^{2} + 3r^{2}\right)\; t + r\;\left(1-6t^{2}\right)\;\; \left(\mathrm{R}-D'\right)\right]\times \\ \times\, \ell^{-1}\left(r,\psi,\,\mathrm{R}-D'\right) - \left[\left(\mathrm{R}^{2}+3r^{2}\right)\; t +\; r\mathrm{R}\;\left(1-6t^{2}\right)\right]\ell^{-1}\left(r,\psi,\,\mathrm{R}\right) + \\ + r\;\left(3t-1\right)\;\ln\left|\frac{\mathrm{R}-D'-r\,t+\ell\;\left(r,\psi,\,\mathrm{R}-D'\right)}{\mathrm{R}-r\,t+\ell\left(r,\psi,\,\mathrm{R}\right)}\right|, \tag{5}$$

where $\ell(r,\psi,\,\mathrm{R}-D')$ and $\ell(r,\psi,\,\mathrm{R})$ are given by

$$\ell\left(r,\psi,\,\mathrm{R}-D'\right) = \sqrt{r^{2} + \left(\mathrm{R}-D'\right)^{2} - 2\,r\;\left(\mathrm{R}-D'\right)\; t}\quad, \tag{6}$$

$$\ell\left(r,\psi,\,\mathrm{R}\right) = \sqrt{r^{2} + \mathrm{R}^{2} - 2\mathrm{R}\,r\,t}. \tag{7}$$

A weak singularity of Newton's integral kernel for $\psi \to 0$ can, for instance, be solved by finding the closed analytical solution for the inner-zone integration domain according to the procedure described by *Sjöberg (2009).*

4. Spectral form

To define the solution of Eq. (1b) in spectral domain, the fundamental harmonic function ℓ^{-1} for the external convergence domain $r \geq \mathrm{R}$ (and $r' < \mathrm{R}$) is presented in the following form (e.g., *Heiskanen and Moritz, 1967*)

$$\ell^{-1}(r, \psi, r') = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^n P_n(t), \tag{8}$$

where $\mathrm{P_n}$ is the Legendre polynomial of degree n. From Eq. (8), the radial derivative of ℓ^{-1} is found to be

$$\frac{\partial \ell^{-1}(r, \psi, r')}{\partial r} = -\frac{1}{r'^2} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^{n+2} (n+1) \; \mathrm{P_n}(t) \; . \tag{9}$$

The substitution from Eq. (9) to Eq. (2) yields

$$K(r, \psi, r') = \int_{r'=\mathrm{R}-D'}^{\mathrm{R}} \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^{n+2} (n+1) \; \mathrm{P_n}(t) \, \mathrm{d}r' \; . \tag{10}$$

Solving the integral of K in Eq. (10), we get

$$K(r, \psi, r') = r \sum_{n=0}^{\infty} \left(\frac{r'}{r} \right)^{n+3} \frac{n+1}{n+3} \mathrm{P_n}(t) \Bigg|_{r'=\mathrm{R}-D'}^{\mathrm{R}} . \tag{11}$$

Substituting for the integral limits in Eq. (11), we arrive at

$$\begin{aligned} K(r, \psi, r') &= \sum_{n=0}^{\infty} \left(\frac{1}{r} \right)^{n+2} \frac{n+1}{n+3} \left[\mathrm{R}^{n+3} - (\mathrm{R} - D')^{n+3} \right] \mathrm{P_n}(t) = \\ &= r \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r} \right)^{n+3} \frac{n+1}{n+3} \left[1 - \left(1 - \frac{D'}{\mathrm{R}} \right)^{n+3} \right] \mathrm{P_n}(t). \end{aligned} \tag{12}$$

The term $(1 - D'/\mathrm{R})^{n+3}$ on the right-hand side of Eq. (12) is further expressed by means of the binomial theorem as follows

$$\left(1-\frac{D'}{\mathrm{R}}\right)^{n+3} \cong \sum_{k=0}^{n+3}\binom{n+3}{k}\frac{(-1)^k}{\mathrm{R}^k}D'^k. \tag{13}$$

After substituting from Eq. (13) to Eq. (12), the spectral representation of K is found to be

$$K\left(r,\psi,r'\right) = -r\sum_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+3}\frac{n+1}{n+3}\sum_{k=1}^{n+3}\binom{n+3}{k}\frac{(-1)^k}{\mathrm{R}^k}D'^k\,\mathrm{P}_\mathrm{n}\left(t\right). \tag{14}$$

From Eqs. (14) and (3), we have

$$\begin{aligned}\delta g^m\left(r,\Omega\right) = {}& \mathrm{G}\sum_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+2}\left(n+1\right)\iint\limits_{\Omega'\in\Phi}\delta\Delta\rho^{\mathrm{c/m}}\left(\Omega'\right)\,D'\left(\Omega'\right)\,\mathrm{P}_\mathrm{n}\left(t\right)\,\mathrm{d}\Omega' - \\ & -\mathrm{G}\,\mathrm{R}\sum_{n=0}^{\infty}\left(\frac{\mathrm{R}}{r}\right)^{n+2}\frac{n+1}{n+3}\sum_{k=2}^{n+3}\binom{n+3}{k}\frac{(-1)^k}{\mathrm{R}^k}\times \\ & \times\iint\limits_{\Omega'\in\Phi}\delta\Delta\rho^{\mathrm{c/m}}\left(\Omega'\right)\,D'^k\left(\Omega'\right)\,\mathrm{P}_\mathrm{n}\left(t\right)\,\mathrm{d}\Omega'.\end{aligned} \tag{15}$$

Since the expansion of the integral kernel K into a series of spherical functions converges uniformly for the external domain $r > \mathrm{R}$, the interchange of summation and integration in Eq. (15) was permitted.

We define the spherical Moho density-depth function M_n of degree n as

$$\begin{aligned}M_\mathrm{n}\left(\Omega\right) &= \frac{2n+1}{4\pi}\iint\limits_{\Omega'\in\Phi}\Delta\rho^{\mathrm{c/m}}\left(\Omega'\right)\,D\left(\Omega'\right)\,\mathrm{P}_\mathrm{n}\left(t\right)\,\mathrm{d}\Omega' = \\ &= \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}}M_{\mathrm{n,m}}\,\mathrm{Y}_{\mathrm{n,m}}\left(\Omega\right).\end{aligned} \tag{16}$$

The corresponding higher-order terms $\left\{\, M_\mathrm{n}^{(\mathrm{k})}:\ k = 2,\,3,\,4,\,...\right\}$ read

$$\begin{aligned}M_\mathrm{n}^{(\mathrm{k})}\left(\Omega\right) &= \frac{2n+1}{4\pi}\iint\limits_{\Omega'\in\Phi}\Delta\rho^{\mathrm{c/m}}\left(\Omega'\right)\,D^k\left(\Omega'\right)\,\mathrm{P}_\mathrm{n}\left(t\right)\,\mathrm{d}\Omega' = \\ &= \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}}M_{\mathrm{n,m}}^{(\mathrm{k})}\,\mathrm{Y}_{\mathrm{n,m}}\left(\Omega\right).\end{aligned} \tag{17}$$

The spherical coefficients $M_{\mathrm{n,m}}$ describe globally the Moho geometry scaled by the Moho density contrast. The same definitions are given for the correction term $\delta\Delta\rho^{\mathrm{c/m}}$. Hence

$$\delta M_{\mathrm{n}}^{(\mathrm{k})}(\Omega) = \frac{2n+1}{4\pi} \iint\limits_{\Omega' \in \Phi} \delta\Delta\rho^{\mathrm{c/m}}(\Omega')\; D^k(\Omega')\; \mathrm{P_n}(t)\; \mathrm{d}\Omega' = \\ = \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \delta M_{\mathrm{n,m}}^{(\mathrm{k})} \, \mathrm{Y_{n,m}}(\Omega) \quad (k = 1,\, 2,\, 3,\, ...)\,. \tag{18}$$

Substitution from Eq. (18) back to Eq. (15) yields

$$\delta g^m(r,\Omega) = 4\pi\,\mathrm{G} \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+2} \frac{n+1}{2n+1} \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \delta M_{\mathrm{n,m}} \, \mathrm{Y_{n,m}}(\Omega) - \\ -\; 4\pi\,\mathrm{G}\,\mathrm{R} \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+2} \frac{1}{2n+1} \frac{n+1}{n+3} \times \\ \times \sum_{k=2}^{n+3} \binom{n+3}{k} \frac{(-1)^k}{\mathrm{R}^k} \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \delta M_{\mathrm{n,m}}^{(\mathrm{k})} \, \mathrm{Y_{n,m}}(\Omega). \tag{19}$$

To relate the spherical functions M_{n} and δM_{n} (and their higher-order terms) with spherical harmonics, which describe the Earth's gravity field, the constituents on the right-hand side of Eq. (19) are scaled by the geocentric gravitational constant $\mathrm{GM} = 3986005 \times 10^8\ \mathrm{m^3\ s^{-2}}$.

For the spherical approximation, the geocentric gravitational constant is given by (e.g., *Novák, 2010*)

$$\mathrm{GM} = \frac{4\pi}{3}\,\mathrm{G}\,\mathrm{R}^3\ \bar{\rho}^{\mathrm{Earth}}, \tag{20}$$

where $\bar{\rho}^{\mathrm{Earth}} = 5500\ \mathrm{kg\ m^{-3}}$ is the Earth's mean mass density.

Combining Eqs. (19) and (20) and limiting the spectral resolution up to the maximum degree $\bar{n}$ of spherical harmonics, we arrive at

$$\delta g^m(r,\Omega) = -\frac{\mathrm{GM}}{\mathrm{R}^2} \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+2} \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} F_{\mathrm{n,m}}^{\delta M} \, \mathrm{Y_{n,m}}(\Omega). \tag{21}$$

The numerical coefficients $F_{\mathrm{n,m}}^{\delta M}$ in Eq. (21) read

$$F_{\mathrm{n,m}}^{\delta M} = \frac{1}{2n+1}\,\frac{3}{\bar{\rho}^{\mathrm{Earth}}}\frac{n+1}{n+3}\sum_{k=1}^{n+3}\binom{n+3}{k}\frac{(-1)^k}{\mathrm{R}^{k+1}}\delta M_{\mathrm{n,m}}^{(\mathrm{k})}\,. \tag{22}$$

The system of observation equations is formed for the correction terms $\delta \mathrm{M}_{\mathrm{n,m}}$ according to Eq. (21). The solution is carried out iteratively using, for instance, a condition of the convergence between results of two successive steps (k and k+1) as follows: $\left\|\delta \mathrm{M}_{\mathrm{n,m}}^{\mathrm{k+1}} - \delta \mathrm{M}_{\mathrm{n,m}}^{\mathrm{k}}\right\|_2 \leq c$, where c is a limit of convergence.

5. Discussion

Tenzer et al. (2012c) estimated that the relative errors in the computed values of δg^{cs} could reach as much as 10% mainly due to large uncertainties of currently available global crust structure models. Since these errors propagate proportionally to the Moho density contrast errors, the same relative uncertainties can be expected in the estimated values of the Moho density contrast especially over areas with variable crustal density structures. Large errors in the estimated values of the Moho density contrast are also expected due to uncertainties within the Moho geometry. Most of the errors in the Moho depth data are linearly related with the errors in the Moho density contrast. These errors propagate to the computed values of δg^m (and subsequently to $\delta \mathrm{M}_{\mathrm{n,m}}$) through uncertainties of the coefficients $D_{\mathrm{n,m}}$ (see Eq. B4), which are generated from discrete values of the Moho depths.

Grad et al. (2009), for instance, demonstrated that the Moho depths uncertainties (estimated based on processing the seismic data) under the Europe regionally exceed 10 km with the average error of more than 4 km. Much larger Moho depth uncertainties are expected over large parts of the world where the seismic data are absent or insufficient. Additional errors in the estimated Moho density contrast are due to the unmodeled gravitational signal from the variable density structures within the mantle lithosphere and sub-lithosphere mantle and eventually also from the geometry of the core-mantle interface. If known, the gravitational signal of anomalous density structures within the mantle should be treated in the same way as the crustal density structures. This can be done by applying the additional gravity stripping correction of the anomalous mantle density structures. However, our current knowledge of spatial mantle density structures is restricted by the lack of reliable global data. A possible way how to partially overcome this problem is to remove the long-wavelength spherical harmonic terms from the isostatic gravity field. The principle of this procedure is based on finding the representative depth of gravity signal attributed to each spherical harmonic degree term (cf. *Eckhardt, 1983*). The spherical harmonics which have the depth below a certain limit (chosen, for instance, as the maximum Moho depth) are removed from the gravity field. Nonetheless, the complete subtraction of the gravity signal of mantle density structures using this procedure is still questionable, because there is hardly any unique spectral distinction between the long-wavelength gravity signal from the mantle and the expected higher-frequency signal of the Moho geometry.

Sjöberg and Bagherbandi (2011) applied the least-square method to simultaneously estimate the Moho depth and density contrast globally using the gravimetric and seismic models. *Tenzer et al. (2012a, 2012b)* used to same method to determine the Moho density contrast under oceans and continents.

6. Conclusions

The gravimetric inverse problem for finding the Moho density contrast was formulated by means of a Fredholm integral equation of the first kind. This method utilizes the direct functional relation between the isostatic gravity data and the Moho density contrast. The observation equations were described using the spatial and spectral representations of the integral kernel.

The gravimetric inverse model directly incorporated the seismic models into the solution. This was done through the gravimetric forward modeling of the gravity data corrected for the gravitational contributions of the topographic and anomalous density structures within the Earth's crust. Moreover, the information on the Moho depths was required in forming the observation equations, which define the relation between the isostatic gravity data and the Moho density contrast.

A principal theoretical advantage of this method is that the relation between the unknown (and sought) Moho density contrast and isostatic gravity disturbances can be readily reformulated for all known Earth's density structures. The realistic estimation of the Moho density contrast is possible only when global crust (and essentially also mantle) density model is available with a sufficient accuracy.

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Appendix A: Consolidated crust-stripped gravity disturbances

The consolidated crust-stripped gravity disturbances δg^{cs} are obtained from the corresponding gravity disturbances δg after applying the topographic and crust density contrasts stripping gravity corrections. The computation is realized according to the following scheme (*Tenzer et al., 2012d*)

$$\delta g^{cs} = \delta g - g^t + g^b + g^i + g^s + g^c, \tag{A.1}$$

where g^t, g^b, g^i, g^s, and g^c are, respectively, the gravitational attractions generated by the topography and density contrasts of the ocean (bathymetry), ice, sediments and remaining anomalous density structures within the consolidated crystalline crust. The spectral representation of δg^{cs} reads (*ibid.*)

$$\delta g^{cs}(r,\Omega) = \frac{\mathrm{GM}}{\mathrm{R}^2} \sum_{n=0}^{\bar{n}} \sum_{m=-n}^{n} \left(\frac{\mathrm{R}}{r}\right)^{n+2} (n+1)\ \mathrm{T}^{\mathrm{cs}}_{\mathrm{n,m}}\, Y_{\mathrm{n,m}}(\Omega), \tag{A.2}$$

where $\mathrm{T}^{\mathrm{cs}}_{\mathrm{n,m}}$ are the coefficients of the consolidated crust-stripped disturbing potential. These coefficients are computed as follows

$$\mathrm{T}^{\mathrm{cs}}_{\mathrm{n,m}} = \mathrm{T}_{\mathrm{n,m}} - \mathrm{V}^{\mathrm{t}}_{\mathrm{n,m}} + \mathrm{V}^{\mathrm{b}}_{\mathrm{n,m}} + \mathrm{V}^{\mathrm{i}}_{\mathrm{n,m}} + \mathrm{V}^{\mathrm{s}}_{\mathrm{n,m}} + \mathrm{V}^{\mathrm{c}}_{\mathrm{n,m}}, \tag{A.3}$$

where $\mathrm{T}_{\mathrm{n,m}}$ are the (fully normalized) numerical coefficients which describe the disturbing potential T (i.e., difference between the Earth's gravity potential W and the normal gravity potential U); and $\mathrm{V}^{\mathrm{t}}_{\mathrm{n,m}}$, $\mathrm{V}^{\mathrm{b}}_{\mathrm{n,m}}$, $\mathrm{V}^{\mathrm{i}}_{\mathrm{n,m}}$, $\mathrm{V}^{\mathrm{s}}_{\mathrm{n,m}}$ and $\mathrm{V}^{\mathrm{c}}_{\mathrm{n,m}}$ are, respectively, the gravitational potential coefficients of topography and density contrasts of the ocean, ice, sediments and consolidated crystalline crust.

The coefficients $V_{\mathrm{n,m}}$ in Eq. (A.3) for the topography and crust density contrasts components read

$$V_{\mathrm{n,m}} = \frac{3}{2n+1} \frac{1}{\bar{\rho}^{\mathrm{Earth}}} \sum_{i=0}^{I} \left(Fl^{(\mathrm{i})}_{\mathrm{n,m}} - Fu^{(\mathrm{i})}_{\mathrm{n,m}}\right). \tag{A.4}$$

The numerical coefficients $\{Fl^{(\mathrm{i})}_{\mathrm{n,m}}, Fu^{(\mathrm{i})}_{\mathrm{n,m}} :\ i = 0, 1, ..., I\}$ are defined as follows

$$Fl^{(\mathrm{i})}_{\mathrm{n,m}} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1+i} \frac{L^{(k+1+i)}_{\mathrm{n,m}}}{\mathrm{R}^{\mathrm{k+1}}}, \tag{A.5}$$

and

$$Fu^{(\mathrm{i})}_{\mathrm{n,m}} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1+i} \frac{U^{(k+1+i)}_{\mathrm{n,m}}}{\mathrm{R}^{\mathrm{k+1}}}. \tag{A.6}$$

The terms $\sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \mathrm{L}_{\mathrm{n,m}}\, \mathrm{Y}_{\mathrm{n,m}}$ and $\sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \mathrm{U}_{\mathrm{n,m}}\, \mathrm{Y}_{\mathrm{n,m}}$ in Eqs. (A.5) and (A.6) define the spherical lower-bound and upper-bound laterally distributed radial density variation functions L_{n} and U_{n} of degree n. These spherical functions and their higher-order terms $\{L_{\mathrm{n}}^{(\mathrm{k+1+i})}, U_{\mathrm{n}}^{(\mathrm{k+1+i})} : \; k = 0, 1, ... ; \; i = 1, 2, ..., I\}$ are defined as follows

$$L_{\mathrm{n}}^{(k+1+i)}(\Omega) = \begin{cases} \frac{4\pi}{2n+1} \iint_{\Phi} \rho(D_U, \Omega')\; D_L^{k+1}(\Omega')\, P_n(t)\; \mathrm{d}\Omega' = \\ = \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i = 0 \\ \\ \frac{4\pi}{2n+1} \iint_{\Phi} \beta(\Omega')\; a_i(\Omega')\; D_L^{k+1+i}(\Omega')\, P_n(t)\; \mathrm{d}\Omega' = \\ = \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1+i)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i = 1, 2, ..., I \end{cases} \tag{A.7}$$

and

$$U_{\mathrm{n}}^{(k+1+i)}(\Omega) = \begin{cases} \frac{4\pi}{2n+1} \iint_{\Phi} \rho(D_U, \Omega')\; D_U^{k+1}(\Omega')\, P_n(t)\; \mathrm{d}\Omega' = \\ = \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i = 0 \\ \\ \frac{4\pi}{2n+1} \iint_{\Phi} \beta(\Omega')\; a_i(\Omega')\; D_U^{k+1+i}(\Omega')\, P_n(t)\; \mathrm{d}\Omega' = \\ = \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1+i)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i = 1, 2, ..., I \end{cases} \tag{A.8}$$

For a specific volumetric layer, the mass density ρ is either constant ρ, laterally-varying $\rho(\Omega')$ or – in the most general case – approximated by the laterally distributed radial density variation model using the following polynomial function (for each lateral column)

$$\rho(r', \Omega') = \rho(D_U, \Omega') + \beta(\Omega') \sum_{i=1}^{I} a_i(\Omega')\; (\mathrm{R} - r')^i,$$
$$\text{for} \quad \mathrm{R} - D_U(\Omega') \geq r' > \mathrm{R} - D_L(\Omega'), \tag{A.9}$$

where $\rho(D_U, \Omega')$ is the nominal value of the lateral density stipulated at the depth D_U of the upper bound of the volumetric mass layer. This density distribution model describes the radial density variation within the volumetric mass layer at the location Ω'. Alternatively, when modeling the gravitational field of the anomalous mass density structures within the Earth's crust, the density contrast $\Delta\rho\,(r',\Omega')$ of the volumetric mass layer relative to the reference crustal density ρ^{c} is defined as

$$\Delta\rho\,(r',\Omega') = \rho^{\mathrm{c}} - \rho\,(r',\Omega') = \Delta\rho\,(D_U,\Omega') - \beta\,(\Omega')\,\sum_{i=1}^{I} a_i\,(\Omega')\,\left(\mathrm{R} - r'\right)^i,$$

$$\text{for}\quad \mathrm{R} - D_U\,(\Omega') \geq r' > \mathrm{R} - D_L\,(\Omega')\,, \tag{A.10}$$

where $\Delta\rho(D_U,\Omega')$ is the nominal value of the lateral density contrast.

The coefficients $\mathrm{L}_{\mathrm{n,m}}$ and $\mathrm{U}_{\mathrm{n,m}}$ combine the information on the geometry and density (or density contrast) distribution of volumetric layer. The coefficients $\mathrm{L}_{\mathrm{n,m}}$ and $\mathrm{U}_{\mathrm{n,m}}$ are generated to a certain degree of spherical harmonics using the discrete data of the spatial density distribution (i.e., typically provided by means of density, depth, and thickness data) of a particular structural component of the Earth's interior. Since the summation in Eq. (A.7) is finite, the validation of the expressions for computing the gravitational field quantities is not restricted to the outer space of the Brillouin sphere. We note that the expressions in Eqs. (A.7) and (A.8) can directly be used if the volumetric mass layer is distributed above and below the sphere of radius R with only one set of the coefficients $\mathrm{L}_{\mathrm{n,m}}$ and $\mathrm{U}_{\mathrm{n,m}}$ for describing the geometry of the lower and upper bounds of this volumetric layer.

Appendix B: Complete crust-stripped isostatic gravity disturbances

The computation of the complete crust-stripped isostatic gravity disturbances δg^m is based on subtracting the isostatic compensation attraction g^i(i.e., the complete-crust stripped gravity correction) from the consolidated crust-stripped gravity disturbances δg^{cs}. The upper bound of the homogeneous crust (density contrast) layer is given by the geoid surface while the lower bound is identical with the (model) Moho density interface. Then we have

$$\delta g^m\,(r,\Omega) = \delta g^{cs}\,(r,\Omega) - g^i =$$

$$= \frac{\mathrm{GM}}{\mathrm{R}} \sum_{n=0}^{\bar{n}} \sum_{m=-n}^{n} \left(\frac{\mathrm{R}}{r} \right)^{n+2} (n+1) \left(\mathrm{T}^{\mathrm{cs}}_{\mathrm{n,m}} - V^{\mathrm{i}}_{\mathrm{n,m}} \right) Y_{\mathrm{n,m}} (\Omega). \quad \text{(B.1)}$$

The potential coefficients $V^{\mathrm{i}}_{\mathrm{n,m}}$ are given by

$$V^{\mathrm{i}}_{\mathrm{n,m}} \cong \frac{3}{2n+1} \frac{\Delta\rho^{\mathrm{c/m}}}{\bar{\rho}^{\mathrm{Earth}}} F^{\mathrm{Moho}}_{\mathrm{n,m}}, \quad \text{(B.2)}$$

where $F^{\mathrm{Moho}}_{\mathrm{n,m}}$ read

$$F^{\mathrm{Moho}}_{\mathrm{n,m}} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1} \frac{D^{(k+1)}_{\mathrm{n,m}}}{\mathrm{R}^{\mathrm{k+1}}}. \quad \text{(B.3)}$$

The spherical Moho-depth function D_{n} of degree n reads

$$D_{\mathrm{n}} (\Omega) = \frac{2n+1}{4\pi} \iint_{\Phi} D \left(\Omega' \right) \; P_n \left(\cos\psi \right) \, \mathrm{d}\Omega' = \sum_{\mathrm{m=-n}}^{\mathrm{n}} D_{\mathrm{n,m}} \, \mathrm{Y}_{\mathrm{n,m}} (\Omega), \quad \text{(B.4)}$$

and

$$D^{(\mathrm{i})}_{\mathrm{n}} (\Omega) = \frac{2n+1}{4\pi} \iint_{\Phi} D^{i} \left(\Omega' \right) \; P_n \left(\cos\psi \right) \, \mathrm{d}\Omega' = \sum_{\mathrm{m=-n}}^{\mathrm{n}} D^{(\mathrm{i})}_{\mathrm{n,m}} \, \mathrm{Y}_{\mathrm{n,m}} (\Omega), \quad \text{(B.5)}$$

where D is the Moho depth, and $D_{\mathrm{n,m}}$ are the coefficients of the global Moho model.