The boundary integral method for the D.C. geoelectric problem in the 3-layered earth with a prismoid inhomogeneity in the second layer

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Abstract: The paper presents algorithm and numerical results for the boundary integral equations (BIE) method of the forward D.C. geoelectric problem for the three-layered earth which contains the prismoidal body with sloped faces in the second layer. This situation occurs in the sedimentary basins. Although the numerical calculations are more complicated in comparison with faces orthogonal to the x, y, z axes, the generalization to the sloped faces enables treatment of the anomalous fields for the bodies of more general shapes as rectangular prisms. The graphs with numerical results present isoline maps of the perturbing potential as well as the resistivity profiles when the source field is due to the pair of D.C. electrodes at the surface of the earth. Also presented apparent resistivity curves for the Schlumberger array *AMNB* sounding.

Key words: geoelectric potential field theory, boundary integral methods, double-layer potential calculation, solid angle of view, apparent resistivity for laterally inhomogeneous media

1. Introduction

The method of BIE developed in the last 30 years has been shown as a very effective one for solving geoelectric potential fields in the layered medium containing 3D or 2D perturbing body; see e.g. *Lee (1975); Okabe (1981); Hvoždara (1982, 1983); Schulz (1985); Eloranta (1986)*. In our earlier papers *(Hvoždara, 1983, 1984)* we have paid attention to the cases of a uniform exciting electric field, which approximates a telluric field for long periods. These general boundary integral formulae can be easily adopted for the cases

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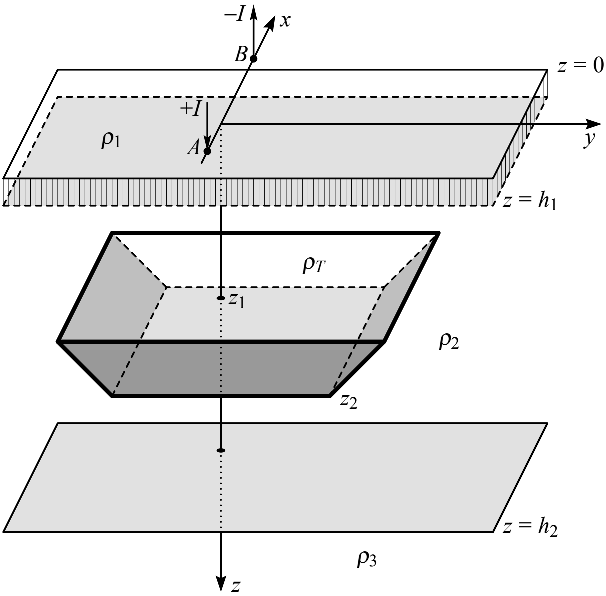


Fig. 1. Model of a 3D disturbing body buried in the second layer of the 3-layered earth.

of non-uniform exciting electric field, which is e.g. due to the point source electrode on the surface of the earth, or by the pair of such electrodes.

In *Hvoždara (1995, 2007)*, we presented detailed extension of the BIE method to the more complicated cases: the 3D body embedded in the superficial layer of 2-layered earth, including its possible contact with the lower or/and upper boundary of the layer, while the source electrode could be situated even on the surface of the outcropping body. The present study is directed into a generalization of our numerical modelling studies to the cases of prismoid bodies bounded by the sloped faces situated in the second layer of a 3-layered (normal) earth, while the body can touch the top or/and bottom layer. This model can also approximate frequent situations occurring in geology, e.g. a depression of the superficial layer into middle layer, or a diapire penetrating from the substratum through the second layer.

2. Boundary integral expressions for potentials and Green's functions calculation

Theoretical formulae for our BIE analysis are similar as those in *Hvoždara*

(1995, 2007), but for better clarity we repeat them also here with necessary modifications. We consider the three-layered earth represented by the superficial layer L_1: $z \in \langle 0, h_1 \rangle$ of resistivity ρ_1, second layer L_2: $z \in \langle h_1, h_2 \rangle$ resistivity ρ_2 and substratum L_3: $z > h_2$ of resistivity ρ_3. In the second layer we consider a 3D disturbing body Ω_T of resistivity ρ_T, bounded by the surface S with a piecewise continuous outer normal $\boldsymbol{n}$ (see Fig. 1). In the absence of disturbing body the D.C. current source excites potentials $V_j(P)$ in the layer $L_j, j = 1, 2, 3$ of resistivity ρ_j. Due to presence of the perturbing body Ω_T these potentials change and result into the total potentials $U_j(P)$. The total potential inside Ω_T is denoted by $U_T(P)$. According to the previous theory presented in *(Hvoždara, 1995, 2007)* we can write expressions for total potentials $U_j(P), U_T(P)$ in the form of sum of unperturbed potentials $V_j(P)$ and generalized double layer potentials (given by the boundary integrals), namely:

$$U_j(P) = V_j(P) + \frac{1}{4\pi} \int_S f(Q) \frac{\partial}{\partial n_Q} G_{j2}(P, Q) \, \mathrm{d}\, S_Q \,, \quad P \in L_j \,, \quad P \notin \Omega_T, \tag{1}$$

$$U_T(P) = \frac{\rho_T}{\rho_2} \left[V_2(P) - v_0 + \frac{1}{4\pi} \int_S f(Q) \frac{\partial}{\partial n_Q} G_{22}(P, Q) \, \mathrm{d}\, S_Q \right] + v_0, \; P \in \Omega_T. \tag{2}$$

Here $G_{j2}(P, Q)$ are Green's functions for the three-layered earth with field calculation point $P \in L_j$ and $Q \in S$. They correspond to the potential of the point source electrode, situated at the point $Q \equiv (x', y', z') \in S$, $z' \in (h_1, h_2)$, calculated for the point $P \equiv (x, y, z) \in L_j$, but instead of source factor $I\rho_2/(4\pi)$ we must put dimensionsless factor equal to 1. The constant v_0 is the mean value of the potential $V_2(P)$ on the surface S. The function $G_{22}(P, Q)$ obeys Poisson equation in the second layer L_2 ($z', z \in \langle h_1, h_2 \rangle$):

$$\nabla^2 G_{22}(P, Q) = -4\pi\delta(P, Q) \,, \quad P \in L_2 \,, \quad Q \in S \,, \tag{3}$$

while $G_{j2}(P, Q)$, $j \neq 2$ are harmonic functions in the layers L_1, L_3, i.e.:

$$\nabla^2 G_{j2}(P, Q) = 0, \quad j = 1, 3. \tag{4}$$

On the surfaces $z = 0$ and $z = h_1, h_2$ there must be satisfied boundary conditions similar to those for the D.C. electric potential due to the point electrode situated in the $Q(x', y', z')$, $z' \in (h_1, h_2)$:

$$[\partial G_{12}(P,Q)/\partial z]_{z=0} = 0\,, \tag{5}$$

$$[G_{12}(P,Q)]_{z=h_1} = [G_{22}(P,Q)]_{z=h_1}\ , \tag{6}$$

$$[\partial G_{12}(P,Q)/\partial z]_{z=h_1} = (\rho_1/\rho_2)\,[\partial G_{22}(P,Q)/\partial z]_{z=h_1}\,. \tag{7}$$

$$[G_{22}(P,Q)]_{z=h_2} = [G_{32}(P,Q)]_{z=h_2}\ , \tag{8}$$

$$[\partial G_{22}(P,Q)/\partial z]_{z=h_2} = (\rho_2/\rho_3)\,[\partial G_{32}(P,Q)/\partial z]_{z=h_2}\,. \tag{9}$$

All functions G_{j2} must have zero limit for $\overline{PQ} \to +\infty$.

The point source singularity of the function $G_{22}(P,Q)$ is expressed by the Dirac function $\delta(P,Q)$ in the equation (3). According to the potential field theory we know that $G_{22}(P,Q)$ must contain the basic singular function $R^{-1} = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{-1/2}$ which obeys the equation

$$\nabla^2(R^{-1}) = -4\pi\delta(P,Q), \tag{10}$$

since R^{-1} is the reciprocal distance of point P from the source point Q. Now we introduce auxiliar cylindrical system (r,φ,z) with polar axis z and

$$r = \left[(x-x')^2 + (y-y')^2\right]^{1/2}, \tag{11}$$

is the horizontal distance from the z-axis. Then the source function R^{-1} can be expressed as:

$$R^{-1} = \left[r^2 + (z-z')^2\right]^{-1/2}. \tag{12}$$

Since the properties of medium are independent of azimuthal angle φ and source function R^{-1} is also of this property, the Laplace equation for Green function $G(r,z)$ is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) + \frac{\partial^2 G}{\partial z^2} = 0, \tag{13}$$

which has general solution in the form

$$G(r,z) = \int\limits_0^\infty \left[Ce^{-tz} + Ee^{tz}\right] J_0(tr)\,\mathrm{d}\,t, \tag{14}$$

where $J_0(tr)$ is the well known Bessel function of the first kind, index zero. In view of the boundary condition (5) we have for the $P \in L_1$ expression $G_{12}(r,z)$

$$G_{12}(r,z) = \int_0^\infty C_1(e^{-tz} + e^{tz}) J_0(tr)\,\mathrm{d}\,t. \tag{15}$$

In the layers L_2, L_3 we have Green's function:

$$G_{22}(r,z) = \frac{1}{R} + \int_0^\infty \left[C_2 e^{-tz} + E_2 e^{tz}\right] J_0(tr)\,\mathrm{d}\,t, \tag{16}$$

$$G_{32}(r,z) = \int_0^\infty C_3 e^{-tz} J_0(tr)\,\mathrm{d}\,t. \tag{17}$$

One can easily find that the boundary condition (5) is satisfied by G_{12} given in (15). For the application of conditions on planar boundaries $z = h_1, h_2$ we have too different expression of the source term $R^{-1} = [r^2 + (z - z')^2]^{-1/2}$ for $z < z'$ and for $z > z'$ by the Weber-Lipschitz integrals:

$$R^{-1} = \int_0^\infty e^{-t(z'-z)} J_0(tr)\,\mathrm{d}\,t, \quad z \in \langle h_1, z' \rangle, \tag{18}$$

$$R^{-1} = \int_0^\infty e^{-t(z-z')} J_0(tr)\,\mathrm{d}\,t, \quad z \in \langle z', h_2 \rangle. \tag{19}$$

Then we can use the boundary conditions (6)–(9) and since they must hold true for all distances r, we obtain system of four linear equations for coefficients C_1, C_2, E_2, C_3:

$$2C_1\,\mathrm{ch}(th_1) = e^{-t(z'-h_1)} + C_2 e^{-th_1} + E_2 e^{th_1}, \tag{20a}$$

$$(\rho_2/\rho_1) 2C_1\,\mathrm{sh}(th_1) = e^{-t(z'-h_1)} - C_2 e^{-th_1} + E_2 e^{th_1}, \tag{20b}$$

$$C_3 e^{-th_2} = e^{-t(h_2-z')} + C_2 e^{-th_2} + E_2 e^{-th_2}, \tag{20c}$$

$$-(\rho_2/\rho_3)C_3e^{-th_2} = -e^{-t(h_2-z')} - C_2e^{-th_2} + E_2e^{th_1}. \tag{20d}$$

We solve this system by the elimination method eliminating first at C_3 from (20c,d) and we obtain first equation for C_2 and E_2:

$$-k_{23}C_2e^{-2th_2} + E_2 = k_{23}e^{-t(2h_2-z')}, \tag{21}$$

where $k_{23} = (1 - \rho_2/\rho_3)/(1 + \rho_2/\rho_3)$. Elimination of C_1 from (20a,b) gives the second equation for C_2, E_2

$$s_{12}C_2e^{-2th_1} + E_2 = -e^{-tz'}, \tag{22}$$

where $$s_{12} = \frac{\mathrm{ch}(th_1) + (\rho_2/\rho_1)\,\mathrm{sh}(th_1)}{\mathrm{ch}(th_1) - (\rho_2/\rho_1)\,\mathrm{sh}(th_1)}. \tag{23}$$

It is possible to adjust the multiplicator s_{12} into form:

$$s_{12} = \frac{1 - k_{12}e^{-2th_1}}{-k_{12} + e^{-2th_1}}, \tag{24}$$

where $k_{12} = (1 - \rho_1/\rho_2)/(1 + \rho_1/\rho_2)$ is the resistivity contrast of layers L_1 and L_2. The solution of Eqs. (21), (22) gives aftersome algebraic operations:

$$C_2 = \Big[k_{23}e^{-t(2h_2-z')} + e^{-tz'} - k_{12}k_{23}e^{-t(2h_2-2h_1-z')} - \\ - k_{12}e^{-t(z'-2h_1)}\Big]\,[F_1(t)]^{-1}\,, \tag{25}$$

$$E_2 = k_{23}\Big[e^{-t(2h_2+z')} + e^{-t(2h_2-z')} - k_{12}e^{-t(2h_2-2h_1+z')} - \\ - k_{12}e^{-t(2h_1+2h_2-z')}\Big]\,[F_1(t)]^{-1}\,. \tag{26}$$

Here we introduced a symbol $F_1(t)$ for the denominator in (25), (26):

$$F_1(t) = 1 - k_{12}e^{-2th_1} - k_{23}e^{-2th_2} + k_{12}k_{23}e^{-2t(h_2-h_1)}. \tag{27}$$

By using Eqs. (20a,b) we easily find the relation:

$$C_1(1 + \rho_2/\rho_1)\left[1 - k_{12}e^{-th_1}\right] = 2e^{-tz'} + 2E_2,$$

which gives after some adjustments with (26) and (27):

$$C_1 = (1 - k_{12}) \left[e^{-tz'} + k_{23} e^{-t(2h_2 - z')} \right] [F_1(t)]^{-1} . \tag{28}$$

Here we have used relation $1 - k_{12} = 2/(1 + \rho_2/\rho_1)$. Equations (20c,d) give relation:

$$C_3 = \frac{2}{1 + \rho_2/\rho_3} \left[e^{tz'} + C_2 \right] ,$$

which after some algebra leads to expression

$$C_3 = (1 + k_{23}) \left\{ e^{-tz'} + \left[k_{23} e^{-t(2h_2 - z')} + e^{-tz'} - \right.\right.$$
$$\left.\left. - k_{12} k_{23} e^{-t(2h_2 - 2h_1 - z')} - k_{12} e^{-t(z' - 2h_1)} \right] [F_1(t)]^{-1} \right\} . \tag{29}$$

Let us note that all coefficients C_1, C_2, E_2, C_3 have the same multiplicator $[F_1(t)]^{-1}$. Because $|k_{12}| < 1$, $|k_{23}| < 1$ and all exponentials are also less than 1, we can expand this factor into infinite geometric series in the manner many times used in D.C. geoelectric theory, see e.g. *Bhattacharya and Patra (1968).* This expansion requires that the depths h_1, h_2 are integer multipliers of some common depth scale D:

$$h_1 = i_1 D, \qquad h_2 = i_2 D, \tag{30}$$

while integer $i_2 > i_1$. Then we obtain series expansion:

$$[F_1(t)]^{-1} = \left[1 - k_{12} e^{-2th_1} - k_{23} e^{-2th_2} + k_{12} k_{23} e^{-2t(h_2 - h_1)} \right]^{-1} =$$
$$= \sum_{n=0}^{\infty} q_n e^{-2nDt}, \tag{31}$$

where $q_0 = 1$ and coefficients q_n can be calculated by the recurrence relations as explained in Appendix I. The coefficients q_n for $n > 1$ are in absolute value less than 1 also. By using expansion (31) we can calculate analytically all integrals of Green's functions $G_{j2}(P, Q)$ given by formulae (15)–(17). All these integrals can be expressed by the infinite sums of Weber-Lipschitz integrals:

$$\int_0^{\infty} e^{-t(2nD + \xi)} J_0(tr) \, \mathrm{d}\, t = \left[r^2 + (2nD + \xi)^2 \right]^{-1/2} , \tag{32}$$

provided the terms $2nD+\xi$ are positive. In this manner we have obtained following formulae:

$$G_{12}(P,Q) = (1-k_{12})^{-1}\left\{(R^{-1}+R_{+}^{-1}) + k_{23}\left[r^2+(2h_2+z-z')^2\right]^{-1/2} + \right.$$
$$\left. + k_{23}\left[r^2+(2h_2-z-z')^2\right]^{-1/2}\right\} + (1-k_{12})\sum_{n=1}^{\infty} q_n \cdot$$
$$\cdot \left\{\left[r^2+(2nD+z+z')^2\right]^{-1/2} + \left[r^2+(2nDh_2-z+z')^2\right]^{-1/2} + \right.$$
$$+k_{23}\left[r^2+(2nD+2h_2+z-z')^2\right]^{-1/2} +$$
$$\left. + k_{23}\left[r^2+(2nD+2h_2-z-z')^2\right]^{-1/2}\right\}. \tag{33}$$

Note that in this function we have

$$R_{+}^{-1} = \left[r^2+(z+z')^2\right]^{-1/2},\ r^2 = (x-x')^2+(y-y')^2,$$

$z' \in (h_1, h_2)$, $z \in \langle 0, h_1\rangle$ and the term R^{-1} is not singular because $z' > h_1$. For the Green's function $G_{22}(P,Q)$ we have formula with singular term R^{-1}, namely:

$$\begin{aligned}
G_{22}(P,Q) = R^{-1} &+ \sum_{n=0}^{\infty} q_n \left[r^2+(2nD+z'+z)^2\right]^{-1/2} + \\
&+ k_{23}\sum_{n=0}^{\infty} q_n \left[r^2+(2nD+2h_2-z'+z)^2\right]^{-1/2} - \\
&- k_{12}\sum_{n=0}^{\infty} q_n \left[r^2+(2nD-2h_1+z'+z)^2\right]^{-1/2} - \\
&- k_{12}k_{23}\sum_{n=0}^{\infty} q_n \left[r^2+(2nD+2h_2-2h_1-z'+z)^2\right]^{-1/2} + \\
&+ k_{23}\left\{\sum_{n=0}^{\infty} q_n \left[\left[r^2+(2nD+2h_2+z'-z)^2\right]^{-1/2} + \right.\right. \\
&\left.\left. + \left[r^2+(2nD+2h_2-z'-z)^2\right]^{-1/2}\right]\right\} - \\
&- k_{12}k_{23}\left\{\sum_{n=0}^{\infty} q_n \left[\left[r^2+(2nD+2h_2-2h_1-z'-z)^2\right]^{-1/2} + \right.\right.
\end{aligned}$$

$$+\left[r^2+(2nD+2h_2-2h_1+z'-z)^2\right]^{-1/2}\Bigg]\Bigg\}. \tag{34}$$

In this function we have $z' \in (h_1, h_2)$ and also $z \in \langle h_1, h_2\rangle$. For the points in L_3 ($z \geq h_2$) we have:

$$\begin{aligned}G_{32}(P,Q) = (1+k_{23})\Bigg\{&R^{-1}+\sum_{n=0}^{\infty} q_n\left[r^2+(2nD+z'+z)^2\right]^{-1/2}+\\&+k_{23}\sum_{n=0}^{\infty} q_n\left[r^2+(2nD+2h_2-z'+z)^2\right]^{-1/2}-\\&-k_{12}\sum_{n=0}^{\infty} q_n\left[r^2+(2nD+2h_2-z'+z)^2\right]^{-1/2}-\\&-k_{12}k_{23}\sum_{n=0}^{\infty} q_n\left[r^2+(2nD+2h_2-h_1-z'+z)^2\right]^{-1/2}\Bigg\}.\end{aligned} \tag{35}$$

Note that the term $R^{-1} = [r^2 + (z - z')^2]^{-1/2}$ is not singular since $z' < z$ in L_3. We can see that expressions for Green's functions contain more terms in comparison with 2-layered earth *(Hvoždara, 1995, 2007)*, but can be also easily adopted for computing.

All three Green's functions occur in integrals of formulae (1)–(2) in the form of their derivatives with respect to the outer normal $\boldsymbol{n}_Q \equiv (n'_x, n'_y, n'_z)$ at the point $Q \in S$, which means:

$$\begin{aligned}&\frac{\partial G_{j2}(P,Q)}{\partial n_Q} \equiv \boldsymbol{n}_Q \cdot \mathrm{grad}_Q\, G_{j2}(P,Q) =\\&=\left(n'_x\frac{\partial}{\partial x'}+n'_y\frac{\partial}{\partial y'}+n'_z\frac{\partial}{\partial z'}\right)G_{j2}(x,y,z;x',y',z')\,, \qquad j=1,2,3\end{aligned} \tag{36}$$

since in Green's functions we have $r^2 = (x - x')^2 + (y - y')^2$, we calculate $\partial G(P,Q)/\partial x' = -(x - x')[r^{-1}\partial G/\partial r]$ and similarly $\partial G(P,Q)/\partial y'$. The form of terms in $G(P,Q)$ guaranties that expression $r^{-1}\partial G/\partial r$ is finite even if $r \to 0$. In formulae (1)–(2) these normal derivatives are integrated, being multiplied by the function $f(Q)$ which represents the density of the double layer distributed over the surface S of the perturbing body Ω_T. This double layer density has to be determined by solving the boundary integral equation which holds true for points $P \in S$:

$$f(P) = 2\beta\,[V_2(P) - v_0] + \frac{\beta}{2\pi} \int\!\!\!\!\!\!\diagdown_S f(Q) \frac{\partial}{\partial n_Q} G_{22}(P,Q)\,\mathrm{d}S_Q\,, \quad P \in S\,, \tag{37}$$

where $\beta = (1 - \rho_2/\rho_T)/(1 + \rho_2/\rho_T)$ and the constant v_0 is

$$v_0 = \frac{1}{|S|} \int_S V_2(P)\,\mathrm{d}\,S_P, \tag{38}$$

(it is the mean value of the exciting potential on the surface S). The BIE (37) is the Fredholm integral equation of the second kind with a weakly singular kernel $K(P,Q) = \partial G_{22}(P,Q)/\partial n_Q$. Its singularity is due to the term $R^{-1} = \left[(x - x')^2 + (y - y')^2 + (z - z')^2\right]^{-1/2}$ in $G_2(P,Q)$. This term becomes singular when $P \to Q$. Fortunately, the surface integral in (37) must be performed in a sense of the principal value (which is denoted by the backslash) and means that a small surface element ΔS_p around the singular point $P \in S$ is excluded from integration. The result reads:

$$\begin{aligned} \int\!\!\!\!\!\!\diagdown_S f(Q) \frac{\partial}{\partial n_Q} G_{22}(P,Q)\,\mathrm{d}S_Q = & \int_{S-\Delta S_p} f(Q) \frac{\partial}{\partial n_Q} (R^{-1})\,\mathrm{d}S_Q + \\ & + \int_S f(Q) \frac{\partial}{\partial n_Q} H_{22}(P,Q)\,\mathrm{d}\,S_Q\,, \end{aligned} \tag{39}$$

where $H_{22}(P,Q) = G_{22}(P,Q) - R^{-1}$ is the non-singular part of the Green's function. The backslash on the integral sign in (37) and (39) denotes integration in the principal value sense. The primary potentials $V_1(P)$ and $V_2(P)$ for a single point electrode supplied with the current I and situated at the surface $z = 0$ can be expressed by the known treatment of D.C. potentials due to a point electrode situated near the surface $z = 0$ of the 3-layer earth at the point $A \equiv (x_A, y_A, z_A)$, $z_A \ll h$. The expressions for $V_1(P)$ and $V_2(P)$ are presented in the Appendix II.

The solution of BIE (37) can be performed analytically only for some simple cases, e.g. spherical body embedded in the unbounded conducting space *(Hvoždara, 1994)*. In that paper we have proved the coincidence of the BIE solution with the solution by means of spherical harmonic functions. The numerical solution is possible by means of a collocation method briefly described in *Hvoždara (1995)*. Let us note that according to *Hvoždara*

(1983) the double-layer density $f(P)$ is in linear relation to the values of the potential $U_T(P)$ on the surface S:

$$f(P) = (1 - \rho_2/\rho_T)[U_T(P) - v_0]\,, \quad P \in S\,. \tag{40}$$

Having solved the BIE (37) we can calculate the potential on the surface of the earth according to the formula (1). Then the electric field is

$$\boldsymbol{E}_1(P) = -\mathrm{grad}\, U_1(P)\,, \tag{41}$$

its components on the surface being:

$$(E_{1x})_{z=0} = -\frac{\partial V_1}{\partial x} - \frac{1}{4\pi}\int_S f(Q)\frac{\partial}{\partial x}\left[\frac{\partial G_{12}(P,Q)}{\partial n_Q}\right]_{z=0} \mathrm{d}S_Q\,, \tag{42}$$

$$(E_{1y})_{z=0} = -\frac{\partial V_1}{\partial y} - \frac{1}{4\pi}\int_S f(Q)\frac{\partial}{\partial y}\left[\frac{\partial G_{12}(P,Q)}{\partial n_Q}\right]_{z=0} \mathrm{d}S_Q\,, \tag{43}$$

while the third component satisfies the well-known boundary condition $(E_{1z})_{z=0} = 0$.

Let us stress that the above formulae are valid when the body Ω_T does not touch with some of its upper planar part S_{h_1} to the boundary $z = h_1$ or with lower boundary S_{h_2} $(z = z_2)$ to the bottom of the layer L_2. Such contact cases must be considered separately as shown in *Hvoždara (1995).*

For the first contact case the body Ω_T touches with its upper planar face $S_{h_1}, (z = z_1 = h_1)$ the top of the layer L_2. There must be considered singularity of normal derivatives of R^{-1} and also $k_{12}R_{h_1}^{-1}$ of $G_{22}(P,Q)$ in the potential $U_T(P)$ given by the (2). The singular term

$$R_{h_1}^{-1} = \left[r^2 + (2h_1 - z - z')^2\right]^{-1/2}, \tag{44}$$

occurs for $n = 0$ in the third sum in formula (34). Since we approach to the surface S_{h_1} from the interior (– side) and $\partial G_{22}/\partial n_Q \equiv -\partial G_{22}/\partial z'$ because on S_{h_1} there is $\boldsymbol{n}_q \equiv (0, 0, -1)$. Then we will find that the limit transition $P \to S_{h_1-}$ gives

$$\lim_{P\to S_{h_1-}} \frac{1}{4\pi}\int_S f(Q)\frac{\partial}{\partial n_Q}(R^{-1} - k_{12}R_{h_1}^{-1})\,\mathrm{d}\,S_Q =$$

$$= -\frac{1}{2}(1 + k_{12})f(P) + \frac{1}{4\pi}\int\!\!\!\!\!\!\diagup_S f(Q)\frac{\partial}{\partial n_Q}(R^{-1} - k_{12}R_{h_1}^{-1})\,\mathrm{d}\,S_Q,\ P \in S_{h_1}. \tag{45}$$

For the second contact case the body Ω_T touches with its lower planar face $S_{h_2}, (z = z_2 = h_2)$ the bottom of the layer L_2. There must be considered singularity of normal derivatives of R^{-1} and also $k_{23}R_{h_2}^{-1}$ of $G_{22}(P,Q)$ in the potential $U_T(P)$ given by the (2). The singular term

$$R_{h_2}^{-1} = \left[r^2 + (2h_2 - z - z')^2\right]^{-1/2}, \tag{46}$$

occurs for $n = 0$ in the fifth sum in formula (34). Since we approach to the surface S_{h_2} from the interior (– side) and $\partial G_{22}/\partial n_Q \equiv \partial G_{22}/\partial z'$ because on S_{h_2} there is $\boldsymbol{n}_q \equiv (0,0,1)$. Then we will find that the limit transition $P \to S_{h_2-}$ gives

$$\lim_{P\to S_{h_2-}} \frac{1}{4\pi}\int_S f(Q)\frac{\partial}{\partial n_Q}(R^{-1} + k_{23}R_{h_2}^{-1})\,\mathrm{d}\,S_Q =$$
$$= -\frac{1}{2}(1-k_{23})f(P) + \frac{1}{4\pi}\int\!\!\!\!\!\!/\!\!/_S f(Q)\frac{\partial}{\partial n_Q}(R^{-1} + k_{23}R_{h_2}^{-1})\,\mathrm{d}\,S_Q,\ P \in S_{h_2}. \tag{47}$$

In this manner we obtain a modified BIE instead of (37):

$$f(P) = 2\gamma\,[V_2(P) - v_0] + \frac{\gamma}{2\pi}\int\!\!\!\!\!\!/\!\!/_S f(Q)\frac{\partial}{\partial n_Q}G_{22}(P,Q)\,\mathrm{d}\,S_Q,\ \text{where} \tag{48}$$

$$\gamma = \begin{cases} \beta \text{ if } P \notin S_{h_1},\ P \notin S_{h_2}, \\ \beta/(1+\beta k_{12}) \text{ if } P \in S_{h_1}, \\ \beta/(1-\beta k_{23}) \text{ if } P \in S_{h_2}. \end{cases}$$

The doubleslash in the integral sign of BIE (48) denotes that for $P \in S_{h_1}$ are omitted contributions of two singular terms $R^{-1} - k_{12}R_{h_1}^{-1}$ and similarly for $P \in S_{h_2}$ contributions from $R^{-1} + k_{23}R_{h_2}^{-1}$, while for the rest of the surface S $(P \notin S_{h_1}, P \notin S_{h_2})$ there are omitted only contributions due to R^{-1}.

3. Numerical calculations and discussion

The crucial part of numerical calculations of BIE method consists of the calculation of integrals with the kernel of type of the double-layer potential

$\boldsymbol{n}' \cdot (\boldsymbol{r}-\boldsymbol{r}') \cdot |\boldsymbol{r}-\boldsymbol{r}'|^{-3}$ over a small subarea ΔS_j which is the part of surface S of the perturbing body Ω_T. For the prismoid with planar faces the basic task is involved in the reliable calculation of such integrals for the triangle planar subarea ΔS_j with corners ABC shown in Fig. 2:

$$\Delta A_j = \int\limits_{\Delta S_j} \frac{\boldsymbol{n}' \cdot (\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \,\mathrm{d}\, S_Q = -\Delta\Omega_j, \tag{49}$$

where $Q(\boldsymbol{r}')$ is the variable point on the subarea ΔS_j. By using of classical knowledge on the Gaussian integral for the double-layer potential, we see that $\Delta\Omega_j$ is the solid angle of view from the point $P(\boldsymbol{r})$ onto planar triangle subarea ΔS_j with outer normal $\boldsymbol{n}' \equiv (n'_x, n'_y, n'_z) \equiv \boldsymbol{n}_Q$. The formula given by *Ivan (1994)* is reliable for the calculation of ΔA_j and we used it in our previous paper *Hvoždara (2007)*. In the further study we found a simpler guide for calculation ΔA_j published in the paper *Guptasarma and Singh (1999)*. We adopted their method with some modifications in the paper *Hvoždara and Majcin (2011)* and also in the present study.

Geometrical situation of the point of view P and triangular subarea with vertices ABC is depicted in Fig. 2. The points $PABC$ form a tetrahedron

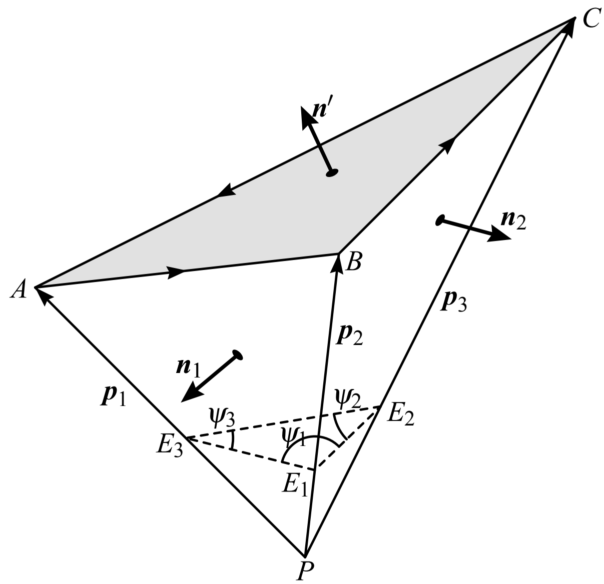


Fig. 2. The parameters for calculation of solid angle of view onto triangular subarea.

with vertice P and triangular base ABC. The position of vertices ABC with respect to the point P is given by vectors $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ while circulation around the triangle is counterclockwise. The normal $\boldsymbol{n}'$ onto the triangular subarea is of unit length and has constant orientation for the whole triangle ABC. Using the basic theory of spherical trigonometry we can calculate the solid angle of view from the point P onto triangular area ABC by means of Girard's formula:

$$\Delta\Omega_j = (\psi_1 + \psi_2 + \psi_3 - \pi) \cdot inp. \tag{50}$$

Here ψ_1 is the inner angle between planes PAB and PBC, ψ_2 is similar angle between PBC and PCA and finally the third angle ψ_3 a is defined for planes PCA and PAB. These angles are depicted on the dashed triangle E_1, E_2, E_3 in Fig. 2. The points E_1, E_2, E_3 are intersections of related vectors $\boldsymbol{p}_i$ with the surface of sphere of unit radius centered in the point P. The number $inp = \pm 1$ is signum of the scalar product $\boldsymbol{p}_1 \cdot \boldsymbol{n}'$. If this scalar product is zero, then $inp = 0$ and $\Delta\Omega_j = 0$ because the point P lies in the plane which contains also the subarea ABC, so the solid angle of view must be zero. Details of calculation of angles ψ_1, ψ_2, ψ_3 by means of scalar and vector products are explained in *Hvoždara and Majcin (2011)* where are mentioned also necessary subroutines SLAGUP3 for triangular subarea and SLAGUP4 for quadrangle subarea.

The numerical calculations were performed in a similar way as in *Hvoždara (2007); Hvoždara and Majcin (2011)* noting that the Green's function $G_{22}(P, Q)$ is now given by the more complicated infinite series (34). Nevertheless, the principal terms are again R^{-1} and $R_{h_1}^{-1} = [r^2 + (2h_1 - z - z')^2]^{-1/2}$, $R_{h_2}^{-1} = [r^2 + (2h_2 - z - z')^2]^{-1/2}$. The special cases when the perturbing body Ω_T touches the bottom and/or upper plane were discussed in Section 2. The BIE (48) can be solved by the collocation method. It means that the surface S of the perturbing body is discretized into M subareas ΔS_j whose centres are denoted as P_m or Q_j. It is also assumed that each subarea is small enough to put $f(Q) = f(Q_j) = \text{const}$ on it. So we introduce the constant approximation of an unknown function $f(Q)$ on ΔS_j. Putting the number M sufficiently large, we can express the BIE (48) in its discretized form:

$$f(P_m) = 2\gamma[V_2(P_m) - v_0] + \sum_{j=1}^{M} f(Q_j) W(P_m, Q_j)\,, \tag{51}$$

where $\gamma = \beta$ if the body does not touch at the point P_m the planar boundary of the surrounding layer and attains modified values as given in (48). The weighting coefficients $W(P_m, Q_j)$ are given by the formula

$$W(P_m, Q_j) = \frac{\gamma}{2\pi} \int\limits_{\Delta S_j} \frac{\partial}{\partial n_Q} G_{22}(P_m, Q) \, \mathrm{d}\, S_Q \, . \tag{52}$$

The integration in the principal value sense was explained in the previous section, and it follows that $W(P_m, Q_j)$ cannot be infinite even if $P_m \equiv Q_m$.

In fact, the formula (51) is the system of M linear equations for the unknown values $f(Q_j)$. This system can be expressed as follows:

$$\sum_{j=1}^{M} \left[\delta_{mj} - W(P_m, Q_j)\right] f(Q_j) = 2\gamma \left[V_2(P_m) - v_0\right] \, , \qquad m = 1, 2, ..., M \, , \tag{53}$$

where δ_{mj} is the Kronecker symbol. This system of equations can be solved using known methods of linear algebra. Once the system (53) is solved, we can calculate the potential and the intensity of the electric field and other geoelectric characteristics, e.g. apparent resistivity.

We checked this algorithm for a 3D perturbing body of the prismoidal block with upper rectangular face at the depth $z_1 \geq h_1$ and $x \in \langle x_{1\ell}, x_{1r} \rangle$, $y \in \langle y_{1\ell}, y_{1r} \rangle$. The bottom face is also rectangular at the depth $z_2 \leq h_2$, $z_2 > z_1$ and $x \in \langle x_{2\ell}, x_{2r} \rangle$, $y \in \langle y_{2\ell}, y_{2r} \rangle$. The block is situated in the second layer of resistivity ρ_2, its thickness being $h_2 - h_1$. The planes of the upper and lower rectangle must be parallel to boundaries $z = h_1, h_2$ and their x, y sides must be parallel each to other. Then the faces of the prismoid connecting the upper and lower rectangle are four planar trapesoids and on each side the vector of normal $\boldsymbol{n'}$ is in fixed direction.

The subdivision of each face was performed by introducing numbers of division (> 5) for edges of each pair of opposite sides of the trapezoid, which is a general form of some face of the prismoid as it was performed in *Hvoždara and Majcin (2011)*. Let us note that for solving the system linear equations (53) for each of the central points P_m we must calculate weighting coefficients $W(P_m, Q_j)$ for all sets of point Q_j, while in Green's function we must treat by using SLAGUP4 at least contributions by terms with R^{-1}, and also from $R_{h_1}^{-1}$, $R_{h_2}^{-1}$. The contributions from other terms in $G_{22}(P, Q)$ given in the series (34) can be calculated by means of central approximation. If we choose the subdivision of each trapezoidal face into 64 quadrangle subareas, we obtain $6 \times 64 = 384 = M$ surface elements ΔS_j, which contribute into summation approximation of the boundary integrals. After solution of linear equation system (53) we obtain $f(Q_j)$ for individual subareas and then we calculate the potential $U_1(P)$ and also the electrical intensity on the surface $z = 0$ for a network of (x, y) points.

We assume that the unperturbed potentials $V_1(P)$ in the layer L_1 and $V_2(P)$ in L_2 are due to the configuration of the $+I$ source electrode at the point $(x_A, 0, z_A)$ and $-I$ electrode at the point $(x_B, 0, z_B)$, $x_B > x_A$, where $z_A, z_B \ll h_1$. Hence, $V_1(P)$ and $V_2(P)$ are expressed by formulae given in the Appendix II. Since we put the source electrodes along the x-axis we use for the A electrode with applied current $+I$ the source factor $q = I\rho_1/(4\pi)$, with the same depths $z_A = z_B = z_0 = h/20$ and for r^2 we use

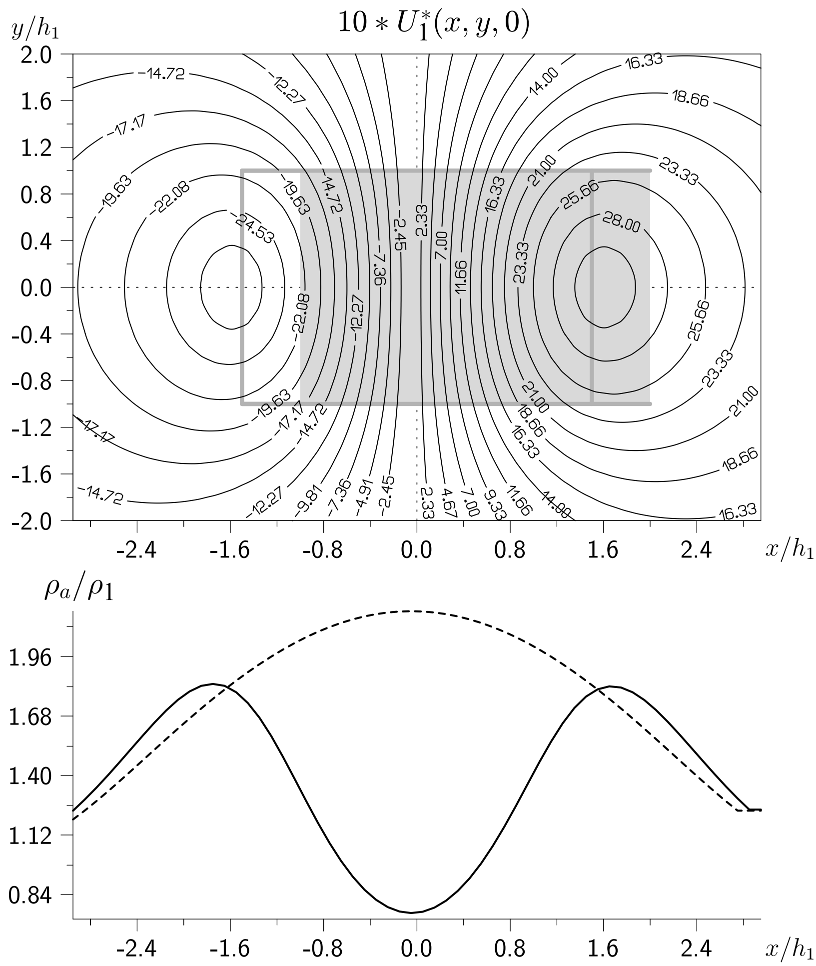
$$r_A^2 = (x - x_A)^2 + y^2, \tag{54}$$

the source point is $Q_A \equiv (x_A, 0, z_A)$. Then unperturbed potentials due to $+I$ electrode in L_1 or L_2 is denoted by $V_{1A}(P, Q_A)$, $V_{2A}(P, Q_A)$, respectively. Similarly, for the unperturbed potentials due to $-I$ electrode situated at the point $Q_B \equiv (x_B, 0, z_B)$ we use for r^2 the expression

$$r_B^2 = (x - x_B)^2 + y^2, \tag{55}$$

and the source factor is $-I\rho_1/(4\pi)$. The unperturbed potentials due to $-I$ electrode is denoted by $V_{1B}(P, Q_B)$, $V_{2B}(P, Q_B)$. Then in the solution of discretized BIE (53) we use for $V_2(P_m)$ the sum $V_{2A}(P_m, Q_A) + V_{2B}(P_m, Q_B)$. After solving the system of linear equations (53) we calculate potential $U_1(P)$ by means of summation approximation of the boundary integral in formula (1) while for $V_1(P)$ we use $V_{1A}(P, Q_A) + V_{1B}(P, Q_B)$. It is clear that the horizontal distance of AB electrodes is $L = |x_B - x_A|$ and they are slightly buried to the depth $z_0 = h_1/20$ in order to avoid large values if we calculate potential $V_1(P)$ at the surface $z = 0$.

The numerical calculations showed that the isoline maps of the potential and electric field components are very similar to those obtained in our previous paper *Hvoždara (2007)* since the dominant field is the unperturbed one in the case 2-layered and for present 3-layered earth. For this reason we present more limited number of figures, whose give characteristics of the perturbed part.



$z_1, xl, xr, yl, yr =$ 1.00 -1.5, 1.5, -1.0, 1.0, m
$z_2, xl, xr, yl, yr =$ 4.00 -1.0, 2.0, -1.0, 1.0, m
$x_A, y_A, z_A =$ -4.00, .00, .05, m
$x_B, y_B, z_B =$ 4.00, .00, .05, m
$h_1, h_2 =$ 1.00, 4.00, m $CU =$ 7.96 V.m
$\rho_1 =$ 100., $\rho_2 =$ 500., $\rho_3 =$ 20., $\rho_T =$ 20. Ω m

Fig. 3a. The isoline map of the anomalous potential $U_1^*(x, y, 0)$ for the prismatic body with parameters given in the table. The profile curves present apparent resistivity ρ_a/ρ_1 (solid curve) for the dipole profiling between source electrodes ($x_A < x < x_B$) and in absence of the prism (dashed curve).

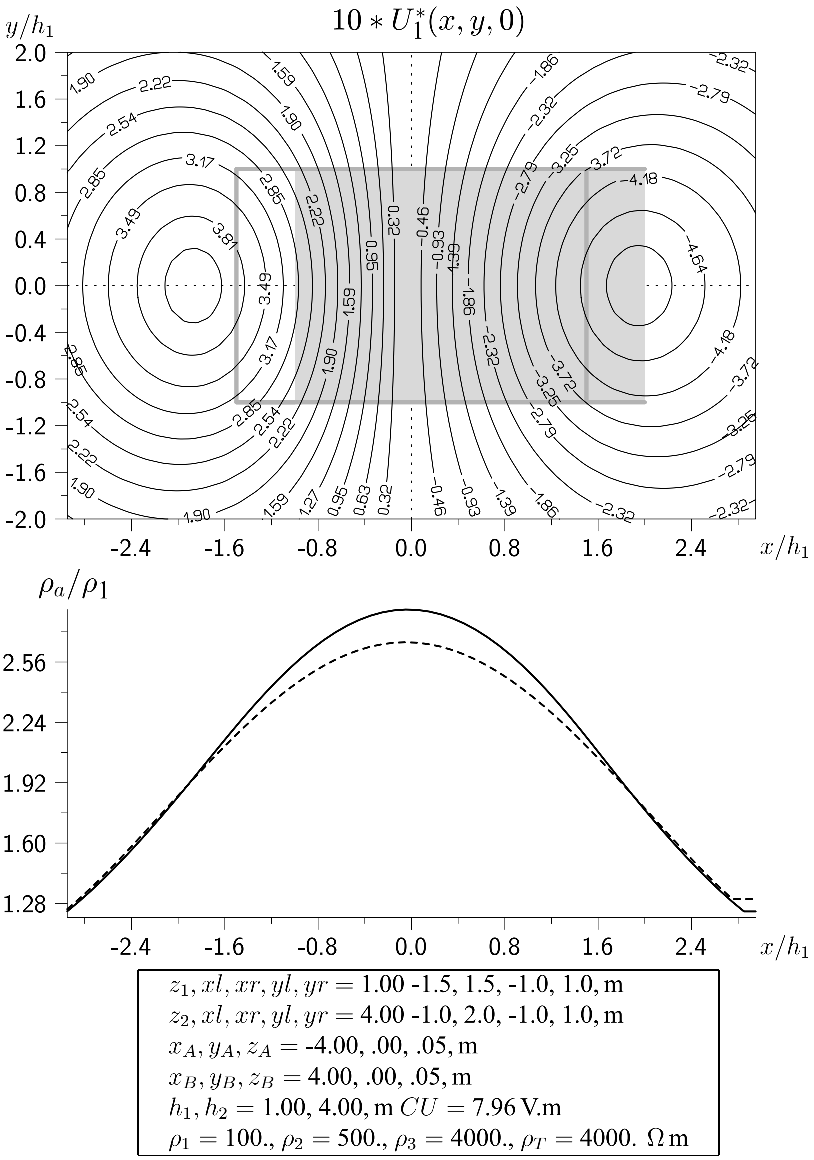


Fig. 3b. The same as in Fig. 3a, but for the high resistive substratum and prism $\rho_T/\rho_2 = 8$, while $\rho_2/\rho_1 = 5$.

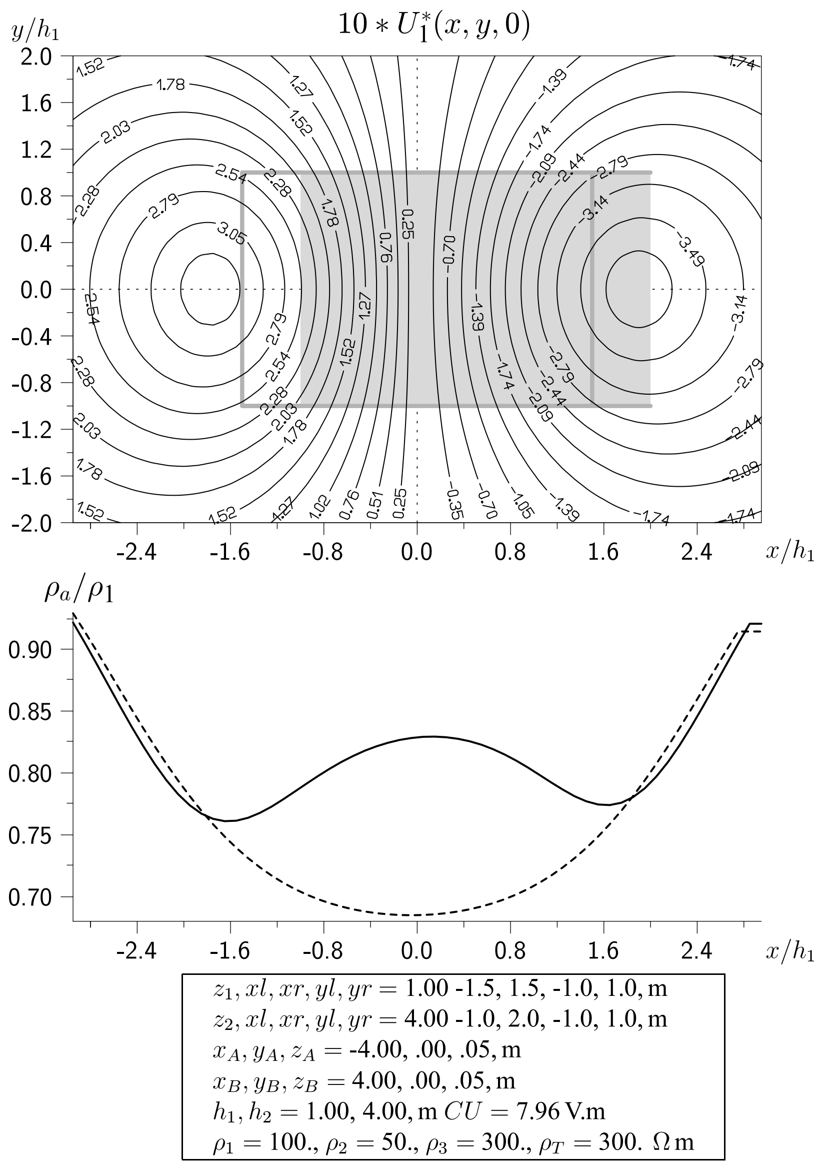


Fig. 3c. The same as in Fig. 3a, but for $\rho_2/\rho_1 = 0.5$ and $\rho_T/\rho_2 = 6$.

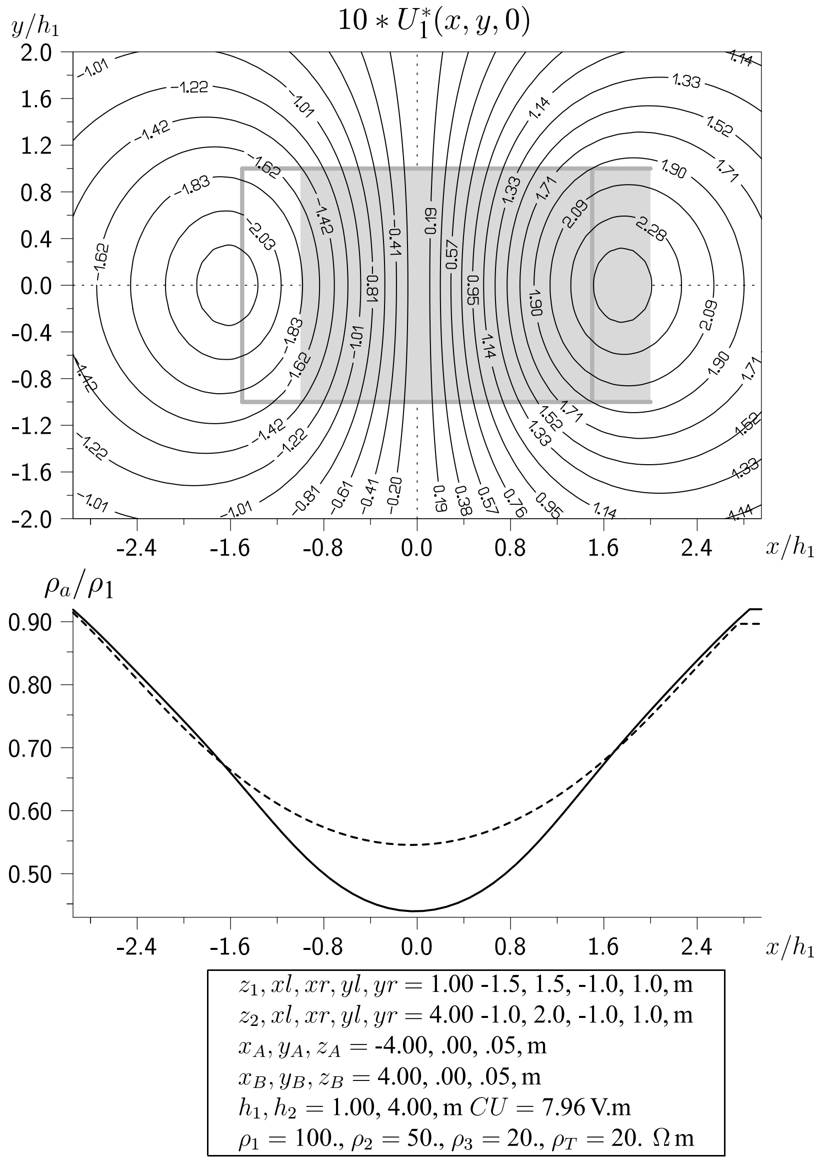
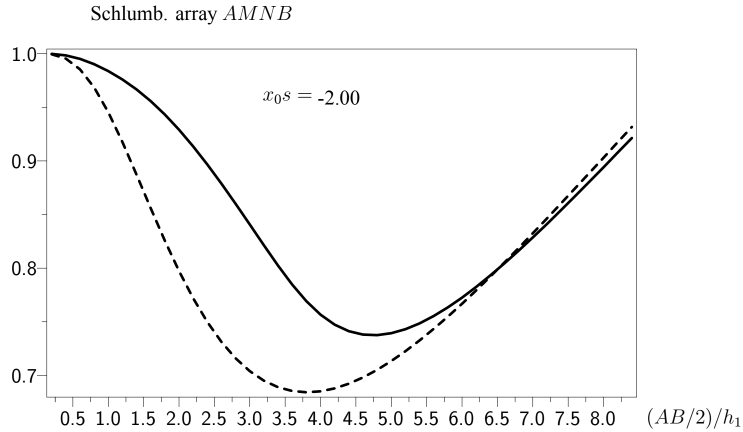


Fig. 3d. The same as in Fig. 3a, but for $\rho_2/\rho_1 = 0.5$ and $\rho_T/\rho_2 = 0.4$.



$z_1, xl, xr, yl, yr =$ 1.00 -1.5, 1.5, -1.0, 1.0, m
$z_2, xl, xr, yl, yr =$ 4.00 -1.0, 2.0, -1.0, 1.0, m
$h_1, h_2, MN =$ 1.00, 4.00, .10, m $CU =$ 7.96 V.m
$\rho_1 =$ 100., $\rho_2 =$ 50., $\rho_3 =$ 300., $\rho_T =$ 300. Ω m

Fig. 4a. The apparent resistivity curves for the prismatic body with parameters given in the table. The centre of Schlumberger array *AMNB* is at the point $x_0s = -2$ m, i.e. outside the left slant prism. The calculated curve ρ_a/ρ_1 (solid curve) is clearly different from the dashed curve in the absence of the prism.

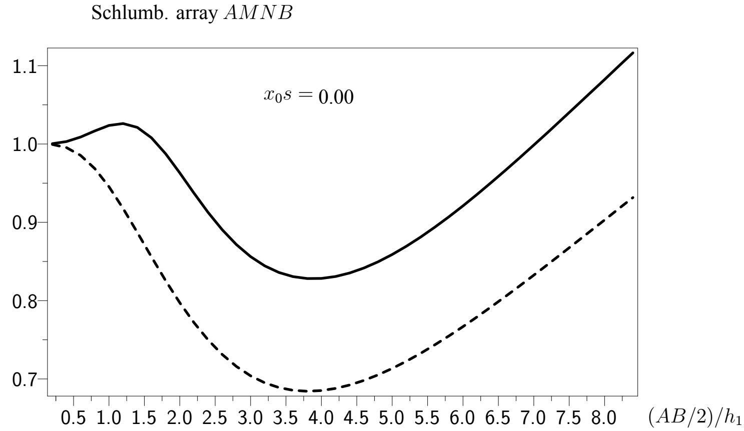
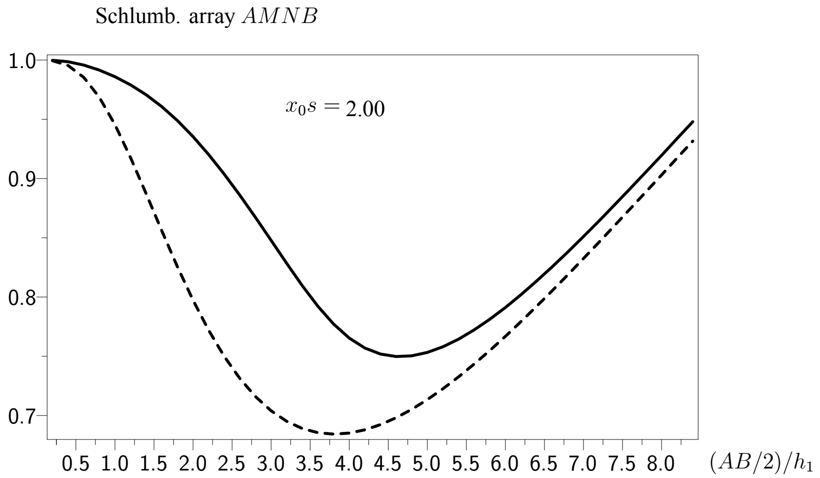


Fig. 4b. The same as in Fig. 4a, but the centre of array is situated at $x_0s = 0$, i.e. above the centre of the upper rectangle of the prism at the depth $z = h_1 = 1$ m. The difference between the solid and dashed curves is more pronounced.



$z_1, xl, xr, yl, yr =$ 1.00 -1.5, 1.5, -1.0, 1.0, m
$z_2, xl, xr, yl, yr =$ 4.00 -1.0, 2.0, -1.0, 1.0, m
$h_1, h_2, MN =$ 1.00, 4.00, .10, m $CU =$ 7.96 V.m
$\rho_1 = 100., \rho_2 = 50., \rho_3 = 300., \rho_T = 300.\ \Omega\,\mathrm{m}$

Fig. 4c. The same as in Fig. 4a, but the centre of array is situated at $x_0s = 2\,\mathrm{m}$, i.e. shifted to the right of the prism.

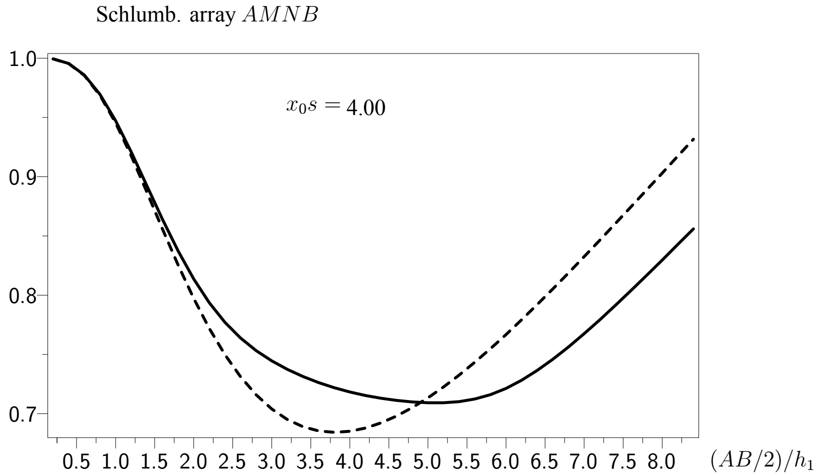


Fig. 4d. The same as in Fig. 4a, but the centre of array is situated at $x_0s = 4\,\mathrm{m}$, i.e. shifted far to the right of the prism.

In our model calculations we put the thickness of the upper layer $h_1 = 1\,\mathrm{m}$, the layer L_2 has bottom plane at the depth $h_2 = 4\,\mathrm{m}$, so its thickness is 3 times that of h_1. We suppose the resistivity of the layer L_1 to be $\rho_1 = 100\,\Omega\mathrm{m}$ and resistivity of the L_2 layer either $\rho_2 = 50\,\Omega\mathrm{m}$ or $\rho_2 = 500\,\Omega\mathrm{m}$, the resistivity of substratum L_3 we put $\rho_3 = 20\,\Omega\mathrm{m}$, or $4000\,\Omega\mathrm{m}$. The resistivity of the prism was assumed to be $\rho_T = \rho_3$, i.e. when $z_1 \geq h_1$ and $z_2 = h_2$ the prism is a 3D dyke penetrating from the substratum L_3 through layer L_2 to the bottom of L_1.

The dimensions of upper and lower rectangle of the prism were 3 m in x-direction and 2 m in y-direction but the bottom rectangle in the depth $z_2 = h_2$ was shifted by 0.5 m to the right in x-direction. In order to have a pronounced effect due to the prism it is clear that we must put quite large distance L between source electrodes, so we put $x_A = -4\,\mathrm{m}$, $x_B = 4\,\mathrm{m}$, then we have penetration depth $AB/2 = 4\,\mathrm{m}$ till the depth h_2 and results are affected by the resistivity ρ_3 of the substratum.

In numerous calculations we realized that most pronounced effect of the perturbing skew prism displays anomalous potential $U_1^*(x, y, 0)$ and also apparent resistivity ρ_a measured on the between fixed source electrodes AB using short voltage dipole MN moved between AB electrode. These results are presented in Figs. 3a–d. The maps of anomalous potential show that it is similar to the potential of buried electric dipole, its polarity is positive (in x-direction) if $\rho_T < \rho_2$, while for $\rho_T > \rho_2$ it is opposite. The presence of the prism is clearly pronounced in the profile curves ρ_a/ρ_1. The comparative curves in the absence of the prism (normal 3-layered earth) are plotted dashed. One can see that apparent resistivity curves for prismatic body (solid curves) are less than dashed, namely in the central region if $\rho_T < \rho_2$ (Figs. 3a, 3d) while for $\rho_T > \rho_2$ the solid curves are greater than normal resistivities (Figs. 3b, 3c). Note that in all Figs. 3a–d in the upper isoline map there are also plotted projections of the upper and lower rectangle of the prism (gray rectangle). One can also see that the geometrical parameters of the model, including fixed positions of AB electrodes are in Figs. 3a–d the same, only resistivities ρ_2, ρ_3, ρ_T are different. Also the current factor $I\rho_1/(4\pi)$ has the same value $CU = 7.96\,\mathrm{Vm}$. For the geoelectric practice it is useful to calculate some models of Schlumberger sounding (*AMNB* array) for our laterally inhomogeneous medium. It is clear that the shape of apparent resistivity curves will change if the centre of the voltage dipole

MN is situated in various positions with respect to the prism. Moreover, the potential is strongly dependent on the mutual position of current electrodes AB with respect to the prismoid. For this purpose we calculated the potential $U_1(P)$ for the case when current electrodes AB are moving along the x-axis, $y = 0$. The centre of voltage electrodes MN is fixed at the ordinate x_0s, while the length of voltage dipole is $MN = 0.1$ m. We present in Figs. 4a–d calculated sounding curves for $x_0s = -2h_1, 0h_1, 2h_1, 4h_1$. The geometrical and resistivity parameters for each sounding curve are given in tables in bottom of figures. It is obvious that the form of the sounding curves strongly depends on both positions of the voltage dipole MN and current electrodes AB. In this manner we consider our modelling calculations as very useful for the better understanding of anomalous geoelectrical fields in the laterally non-uniform media.

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Appendix I

All formulae for coefficients C_1, C_2, E_2, C_3 calculated in Section 2 have the same denominator $F_1(t)$ given by the formula (31). There is a possibility to expand the function $[F_1(t)]^{-1}$ into infinite series if the values h_1, h_2 are integer multiplies of some depth scale D:

$$h_1 = i_1 D, \quad h_2 = i_2 D, \quad i_1, i_2 \text{ integers} \tag{I.1}$$

with $i_2 > i_1$ and $i_1 \geq 1$. Now we introduce notations:

$$g = e^{-2Dt}, \tag{I.2}$$

$$f_1 = k_{12}, \quad f_2 = k_{23}, \quad f_3 = -k_{12}k_{23}, \tag{I.3}$$

then we can express $[F_1(t)]^{-1}$ as follows:

$$\left[1 - k_{12}\, e^{-2th_1} - k_{23}\, e^{-2th_2} + k_{12}k_{23}e^{-2t(h_2-h_1)}\right]^{-1} =$$

$$\left[1 - f_1 g^{i_1} - f_2 g^{i_2} - f_3 g^{i_2-i_1}\right]^{-1} = \sum_{n=0}^{\infty} q_n g^n, \tag{I.4}$$

where $g < 1$ and also f_1, f_2, f_3 are with absolute value less than 1 and q_n are to be determined. The relation (I.4) will be satisfied if

$$1 = \left[1 - f_1 g^{i_1} - f_2 g^{i_2} - f_3 g^{i_2 - i_1}\right] \sum_{n=0}^{\infty} q_n g^n. \tag{I.5}$$

Let us now multiply individual terms in square brackets with the infinite sum, so obtaining:

$$1 = \sum_{n=0}^{\infty} q_n g^n - f_1 \sum_{n=0}^{\infty} q_n g^{n+i_1} - f_2 \sum_{n=0}^{\infty} q_n g^{n+i_2} - f_3 \sum_{n=0}^{\infty} q_n g^{n+i_2-i_1}. \tag{I.6}$$

If we perform a suitable shift of summation indices in second, third and fourth series and compare the resulting series with the unit in l.h.s. in (I.6), we obtain recurence relations for coefficients q_n:

$$\begin{aligned} &q_0 = 1, \\ &q_n = f_1 q_{n-i_1} + f_3 q_{n-i_2+i_1}, \quad \text{for } n = 1, 2, \ldots, n_1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{where } n_1 = \max(i_1, i_2 - i_1) \\ &q_n = f_1 q_{n-i_1} + f_2 q_{n-i_2} + f_3 q_{n-i_2+i_1}, \quad \text{for } n > n_1, \end{aligned} \tag{I.7}$$

while we must take $q_k \equiv 0$ if $k < 0$. Then we can use these coefficients as expansion $[F_1(t)]^{-1}$ into infinite series:

$$[F_1(t)]^{-1} = \sum_{n=0}^{\infty} q_n \, e^{-2nDt}, \tag{I.8}$$

in all integrals with C_1, C_2, E_2, C_3. Numerical calculations show that $|q_n| < 1$ for $n \geq 1$ so the series in (I.8) rapidly converges, while $|k_{12}| < 1$, $|k_{23}| < 1$.

Appendix II

In this appendix we present calculation formulae for the potentials due to D.C. electrode buried in the top layer L_1 of the 3-layered earth. In order to avoid confusion with similar calculations of the Green's functions we decided to present calculations of normal potentials $V_1(P)$, $V_2(P)$, $V_3(P)$ separately. The source electrode we suppose is in the first layer in the point $A \equiv (x_0, y_0, z_0)$ at the depth $z_0 \ll h_1$, with resistivity ρ_1 i.e. in the layer L_1. The primary potential is of the well known type

$$V_0(P) = \frac{I\rho_1}{4\pi} \left(\frac{1}{R_0} + \frac{1}{R_{0+}} \right), \tag{II.1}$$

where $R_0 = \left[r^2 + (z - z_0)^2\right]^{1/2}$, $R_{0+} = \left[r^2 + (z + z_0)^2\right]^{1/2}$, and $r^2 = (x - x_0)^2 + (y - y_0)^2$. This potential satisfies the known boundary condition on the plane $z = 0$:

$$[\partial V_0/\partial z]_{z=0} = 0. \tag{II.2}$$

It is clear that the potentials due to primary potential (II.1) are axially symmetric with respect to the polar axis z in all three layers. The method of separation of variables in the cylindrical system (r, z) gives axially symmetric expressions for potentials in the individual layers L_1, L_2, L_3:

$$V_1(r,z) = q\left\{R_0^{-1} + R_{0+}^{-1} + \int\limits_0^\infty A_1(e^{-tz} + e^{tz})J_0(tr)\,\mathrm{d}\,t\right\}, \tag{II.3}$$

$$V_2(r,z) = q\int\limits_0^\infty \left[A_2 e^{-tz} + B_2 e^{tz}\right] J_0(tr)\,\mathrm{d}\,t, \tag{II.4}$$

$$V_3(r,z) = q\int\limits_0^\infty A_3 e^{-tz} J_0(tr)\,\mathrm{d}\,t, \tag{II.5}$$

where $q = I\rho_1/(4\pi)$ is the source multiplicator and $J_0(tr)$ is Bessel function of the first kind, zero index. It is clear that potential $V_1(r,z)$ satisfies the boundary condition

$$[\partial V_1/\partial z]_{z=0} = 0, \tag{II.6}$$

since this boundary separates non-conducting air and the layer L_1 of finite electrical conductivity $\sigma_1 = \rho_1^{-1}$. On the boundaries $z = h_1, h_2$ there must be satisfied continuity of potentials and vertical density of electric current $j_z = -\rho^{-1}\partial V/\partial z$. We must express the potential $V_0(r,z)$ also as integral with kernel $e^{\pm tz}J_0(tr)$ by using well known Weber-Lipschitz formula:

$$V_0(P) = q\int\limits_0^\infty \left[e^{-t(z-z_0)} + e^{-t(z+z_0)}\right] J_0(tr)\,\mathrm{d}\,t, \quad z > z_0, \tag{II.7}$$

valid for $z \in (z_0, h_1)$. Then the boundary conditions on the planes $z = h_1, h_2$ give the system of linear equations for A_1, A_2, B_2, A_3:

$$\begin{aligned}
e^{-t(h_1-z_0)} + e^{-t(h_1+z_0)} + 2A_1 \operatorname{ch}(th_1) &= A_2 e^{-th_1} + B_2 e^{th_1},\\
-e^{-t(h_1-z_0)} - e^{-t(h_1+z_0)} + 2A_1 \operatorname{sh}(th_1) &= -(\rho_1/\rho_2)\left[A_2 e^{-th_1} - B_2 e^{th_1}\right],\\
A_2 e^{-th_2} + B_2 e^{th_2} &= A_3 e^{-th_2},\\
-A_2 e^{-th_2} + B_2 e^{th_2} &= -(\rho_2/\rho_3) A_3 e^{-th_2}.
\end{aligned} \tag{II.8}$$

This system can be solved by the elimination method. We multiply third equation with ρ_2/ρ_3 and sum with fourth one, then we eliminate A_3 and obtain:

$$-(1-\rho_2/\rho_3)A_2 e^{-th_2} + (1+\rho_2/\rho_3)B_2 e^{th_2} = 0,$$

so we have a relation between B_2 and A_2:

$$B_2 = k_{23} e^{-2th_2} A_2, \tag{II.9}$$

where $k_{23} = (1-\rho_2/\rho_3)/(1+\rho_2/\rho_3)$ is resistivity contrast factor of layers L_2 and L_3. Introducing relation (II.9) into the first twoequations of the system (II.8) we obtain 2 equations for A_1, A_2:

$$2e^{-th_1} \operatorname{ch}(tz_0) + 2A_1 \operatorname{ch}(th_1) = A_2(e^{-th_1} + k_{23}\, e^{-2th_2} e^{th_1}), \tag{II.10}$$

$$-2e^{-th_1} \operatorname{ch}(tz_0) + 2A_1 \operatorname{sh}(th_1) = -\frac{\rho_1}{\rho_2} A_2(e^{-th_1} - k_{23}\, e^{-2th_2} e^{th_1}). \tag{II.11}$$

Now we introduce the coefficient

$$W_{12} = \frac{(\rho_1/\rho_2)\left[e^{-th_1} - k_{23}\, e^{-t(2h_2-h_1)}\right]}{e^{-th_1} + k_{23}\, e^{-t(2h_2-h_1)}}, \tag{II.12}$$

and in the system (II.8) we multiply the first equation with them and sum with the second one giving equation for A_1:

$$A_1 = \frac{(1-W_{12})e^{-th_1} \operatorname{ch}(tz_0)}{W_{12} \operatorname{ch}(th_1) + \operatorname{sh}(th_1)}. \tag{II.13}$$

After some algebra, using W_{12} from (II.12) we obtain final expression for the coefficient A_1:

$$A_1 = \frac{\left[k_{12}\, e^{-2th_1} + k_{23}\, e^{-2th_2}\right] 2\,\mathrm{ch}(tz_0)}{1 - k_{12}\, e^{-2th_1} - k_{23}\, e^{-2th_2} + k_{12}k_{23}\, e^{-2t(h_2-h_1)}}, \tag{II.14}$$

where $k_{12} = (1-\rho_1/\rho_2)/(1+\rho_1/\rho_2)$. By using equation (II.10) we obtain expression for A_2 as follows:

$$A_2 = \left[(1+k_{12})2\,\mathrm{ch}(tz_0)\right]/F_1(t), \tag{II.15}$$

where $F_1(t)$ denotes the denominator of A_1 in (II.14) i.e.:

$$F_1(t) = 1 - k_{12}\, e^{-2th_1} - k_{23}\, e^{-2th_2} + k_{12}k_{23} e^{-2t(h_2-h_1)}. \tag{II.16}$$

The coefficient B_2 can be calculated from (II.9):

$$B_2 = \left[k_{23}(1+k_{12})\, e^{-2th_2} 2\,\mathrm{ch}(tz_0)\right]/F_1(t). \tag{II.17}$$

Finally we determine the coefficient A_3 by using third equation of the system (II.8):

$$A_3 = A_2 + B_2\, e^{2th_2} = \left[(1+k_{12})(1+k_{23})2\,\mathrm{ch}(tz_0)\right]/F_1(t). \tag{II.18}$$

The expressions for A_1, A_2, B_2, A_3 enable calculations of potentials in all three layers.

When we know the analytical formulae for coefficients A_1, A_2, B_2, A_3 in potentials (II.3–II.5) we can calculate numerically these integrals by using some of algorithms for numerical Hankel transform. More effective is application of well known treatment in D.C. geoelectricity for multilayered media. Traditionally the source electrode on surface of the earth (i.e. $z_0 = 0$) and also calculation points on the surface $z = 0$ are considered, which simplifies calculations. Such a treatment is given e.g. in monograph *Bhattacharya and Patra (1968)*. We can use similar treatment for our more general problem with $z_0 \in \langle 0, h_1\rangle$ and $z \geq 0$ distinguishing z-position in three layers.

In the Appendix I we have showed that we can expand the function $[F_1(t)]^{-1}$ into infinite series:

$$[F_1(t)]^{-1} = \sum_{n=0}^{\infty} q_n e^{-2nDt},$$

where D is a common scale for planar boundary depths h_1, h_2.

We can easily calculate integral parts in $V_1(r,z)$ and also $V_2(r,z)$, $V_3(r,z)$. For the potential $V_1(r,z)$ we need a calculate two integrals

$$IA_1^- = \int_0^\infty A_1\, e^{-tz} J_0(tr)\, \mathrm{d}\, t, \tag{II.19}$$

$$IA_1^+ = \int_0^\infty A_1\, e^{tz} J_0(tr)\, \mathrm{d}\, t. \tag{II.20}$$

We use formula (II.14) for A_1 and expansion (I.8), then we obtain expression for IA_1^-:

$$\begin{aligned} IA_1^- = k_{12} \sum_{n=0}^{\infty} q_n & \left\{ \left[r^2 + (2h_1 + 2nD + z_0 + z)^2 \right]^{-1/2} + \right. \\ & + \left. \left[r^2 + (2h_1 + 2nD - z_0 + z)^2 \right]^{-1/2} \right\} + \\ & + k_{23} \sum_{n=0}^{\infty} q_n \left\{ \left[r^2 + (2h_2 + 2nD + z_0 + z)^2 \right]^{-1/2} + \right. \\ & + \left. \left[r^2 + (2h_2 + 2nD - z_0 + z)^2 \right]^{-1/2} \right\}. \end{aligned} \tag{II.21}$$

It is clear that we have used the known Weber–Lipschitz integral:

$$\int_0^\infty e^{-2t\xi} J_0(tr)\, \mathrm{d}\, t = \left[r^2 + \xi^2 \right]^{-1/2}, \quad \xi > 0. \tag{II.22}$$

We can see that a necessary condition for convergence of integrals is satisfied for all terms in (II.21) containing $h_1, h_2,\ z, z_0$ even for $n = 0$. Similarly we have:

$$\begin{aligned} IA_1^+ = k_{12} \sum_{n=0}^{\infty} q_n & \left\{ \left[r^2 + (2h_1 + 2nD + z_0 - z)^2 \right]^{-1/2} + \right. \\ & + \left. \left[r^2 + (2h_1 + 2nD - z_0 - z)^2 \right]^{-1/2} \right\} + \\ & + k_{23} \sum_{n=0}^{\infty} q_n \left\{ \left[r^2 + (2h_2 + 2nD + z_0 - z)^2 \right]^{-1/2} + \right. \\ & + \left. \left[r^2 + (2h_2 + 2nD - z_0 - z)^2 \right]^{-1/2} \right\}. \end{aligned} \tag{II.23}$$

For the potential $V_2(r, z)$ we use formulae (II.15), (II.17) for A_2, B_2 and we obtain:

$$\int_0^\infty A_2 e^{-tz} J_0(tr)\,\mathrm{d}\,t = (1+k_{12}) \sum_{n=0}^\infty q_n \left\{ \left[r^2 + (2nD + z_0 + z)^2\right]^{-1/2} + \right.$$

$$\left. + \left[r^2 + (2nD - z_0 + z)^2\right]^{-1/2} \right\}, \tag{II.24}$$

$$\int_0^\infty B_2 e^{tz} J_0(tr)\,\mathrm{d}\,t = k_{23}(1+k_{12}) \sum_{n=0}^\infty q_n \left\{ \left[r^2 + (2h_2 + 2nD + z_0 - z)^2\right]^{-1/2} + \right.$$

$$\left. + \left[r^2 + (2h_2 + 2nD - z_0 - z)^2\right]^{-1/2} \right\}. \tag{II.25}$$

These series are convergent for $z \in \langle h_1, h_2 \rangle$, $z_0 \in \langle 0, h_1)$. For the potential V_3 we use formula (II.18) for A_3 yielding

$$\int_0^\infty A_3 e^{-tz} J_0(tr)\,\mathrm{d}\,t = (1+k_{12})(1+k_{23}) \cdot$$

$$\cdot \sum_{n=0}^\infty q_n \left\{ \left[r^2 + (2nD + z_0 + z)^2\right]^{-1/2} + \left[r^2 + (2nD - z_0 + z)^2\right]^{-1/2} \right\}. \tag{II.26}$$

Using the series (II.21)–(II.26) we can calculate potentials in all points of the 3-layered medium.We see that the terms in infinite series are of multiple reflection type due to three planar boundaries.