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Gravity field due to a homogeneous oblate spheroid: Simple solution form and numerical calculations

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Abstract: We present a simple derivation of the interior and exterior gravitational potentials due to oblate spheroid and also its gravity field components by using the fundamental solution of the Laplace equation in oblate spheroidal coordinates. Application of the method of separation of variables provides an expression for the potential in terms of oblate spheroidal harmonics of degree $n = 0, 2$. This solution is more concise and suitable for the numerical calculations in comparison with infinite series in spherical harmonics. Also presented are the computations in the form of potential isolines inside and outside the spheroid, as well as for the gravity field components. These reveal some interesting properties of the gravity field of this fundamental geophysical body useful for the applied gravimetry.

Key words: ellipsoid gravity field, oblate ellipsoidal coordinates, oblate ellipsoidal harmonics

1. Introduction

The problem of the gravity potential due to a uniform oblate ellipsoid has been treated by many mathematicians in the last two centuries. It is necessary to recall the classical papers by P. S. Laplace, S. D. Poisson, and G. Green quoted in numerous monographs (e.g. *Hobson, 1931*; *Heiskanen and Moritz, 1967*; *Duboshin, 1961*; *Grushinskyj, 1963*; *Muratov, 1975*; *Pick et al., 1973; Grafarend et al., 2010*). As a rule the formulae quoted in these monographs present only the potential outside the spheroid in order to approximate our planet Earth with very small oblateness. The numerical calculations (if some were prepared) concern also the Earth, including its rotational acceleration potential.

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It is well known that the oblate spheroid is axially symmetric 3D body, which is bounded by the rotation of the generating ellipse with semiaxes a and b $(a > b)$. The rotation axis is assumed to coincide with the vertical minor semi axis b. The oblate spheroid is useful as an approximation to: i) geological anomalous bodies (e.g. laccolithies) with a, b in scales 50–500 m, ii) planetary bodies with dimensions of thousands km and iii) for protoplanetary dust disc with $b \ll a$, while a is about 100 AU ($\approx 150 \times 10^8$ km). For the two latter types of bodies the rotation of the spheroid must be considered.

There exist compact expressions for the exterior potential of the non-rotating oblate spheroid e.g. *Kellog (1929); MacMillan (1958)* based on the expression of the volume integration in the ellipsoidal coordinates, but these need a solution of the auxiliar quadratic equation for every exterior point coordinates x, y, z. This makes some complications, especially if there is a need to calculate derivatives of the potential, which are used in the applied gravimetry. The Newton's gravitational potential of the spheroid calculated for exterior and interior points has been discussed recently by *Wang (1988; 1989)*, where the transformation to the oblate spheroidal harmonics is also shown. The main motivation of our work was a unified derivation of the exterior and interior potentials by using the method of separation of variables in the oblate spheroidal coordinates. This treatment seems to be more concise and straightforward as compared to traditional approaches.

The absence of this general approach and numerical results was the main motivation of our analysis. The application of the method of separation of variables allows for the numerical calculations for a very oblate ellipsoid with $b/a = 0.4$ or for a nearly flat disc with $b/a = 0.1$ to be performed in a straightforward way.

2. Gravity potential due to oblate ellipsoid

The axially symmetric oblate ellipsoid is the body which is bounded by surface of the 2nd degree described by the equation:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1, \tag{1}$$

where $a(b)$ are the major (minor) semiaxes of the ellipsoid, centered in the point $O \equiv (0, 0, 0)$. The section of the ellipsoid boundary by the plane (x, z) is depicted in Fig. 1. The density inside the spheroid is assumed uniform, ρ_0. Theory of the Newton's gravitational potential implies that the potential U_T inside the spheroid volume obeys Poisson's equation:

$$\nabla^2 U_T(x, y, z) = -4\pi G \rho_0, \tag{2}$$

where $G = 6.67428 \times 10^{-11}$ m^{3}kg^{-1}s^{-2} is the gravitational constant. Outside the body τ the potential U_1 is a harmonic function satisfying:

$$\nabla^2 U_1(x, y, z) = 0, \tag{3}$$

i.e. it obeys the Laplace equation. It is clear that both potentials must keep rotational symmetry with respect to the axis z. Hence, the pattern of the potentials U_T, U_1 will be the same as that in the plane (x, z). In a view of

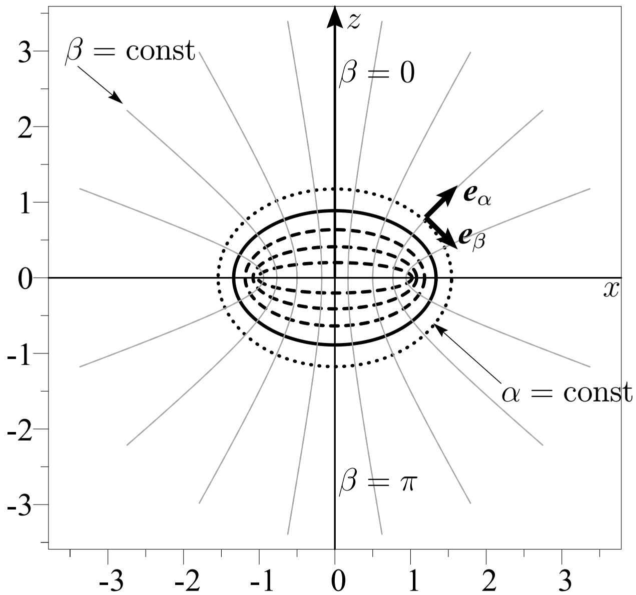


Fig. 1. The (x, z) cross-section of the confocal oblate ellipsoids family $\alpha = \mathrm{const}$ (dashed and dotted lines) generated by the rotation of the basic ellipse (solid line) around the z axis. The curvilinear unit vectors $\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta$ are also depicted at one point. The thin lines depict the orthogonal hyperboloids $\beta = \mathrm{const}$.

this axial symmetry, we can easily prove that the suitable solution of (2) has the form:

$$V_0(x,y,z) = -\pi G\rho_0(x^2+y^2), \quad P\in\tau. \tag{4}$$

Then the potential inside the spheroid τ is given by the sum of the two parts:

$$U_T(x,y,z) = V_0 + U_T^*, \quad P\in\tau, \tag{5}$$

where the additional potential U_T^* is a harmonic function, i.e. it obeys the Laplace equation:

$$\nabla^2 U_T^*(x,y,z) = 0. \tag{6}$$

It is well known that across the surface S of the spheroid the potentials and their normal derivatives must be continuous. To solve this boundary potential problem, for the oblate spheroid it is suitable to transform to the curvilinear orthogonal system (α,β,φ). This transformation is defined in numerous monographs (e.g. *Morse and Feshbach, 1953*; *Lebedev, 1963*; *Arfken, 1966*). The transformation relations between the curvilinear coordinates (α,β,φ) and the Cartesian coordinates (x,y,z) are as follows:

$$\begin{aligned} x &= f\,\mathrm{ch}\,\alpha\ \sin\beta\ \cos\varphi\,, \\ y &= f\,\mathrm{ch}\,\alpha\ \sin\beta\ \sin\varphi\,, \\ z &= f\,\mathrm{sh}\,\alpha\ \cos\beta\,. \end{aligned} \tag{7}$$

The "ellipticity coordinate" $\alpha \in \langle 0\,,+\infty)$, the polar angle $\beta\in\langle 0\,,\pi)$, and the azimuthal angle $\varphi\in\langle 0\,,2\pi)$. The length parameter f is linked to the semiaxes of the generating ellipse by the formula:

$$f = \left(a^2-b^2\right)^{1/2}. \tag{8}$$

By elimination of parameters β and φ from relations (7), we find that the surfaces $\alpha = \mathrm{const}$ are rotational ellipsoids described by the equation:

$$\frac{x^2+y^2}{f^2\,\mathrm{ch}^2\,\alpha} + \frac{z^2}{f^2\,\mathrm{sh}^2\,\alpha} = 1. \tag{9}$$

For the surface S of our spheroid (1) we obtain the associated value of α_0, given by the formulae:

$$f^2 \,\mathrm{ch}^2 \,\alpha_0 = a^2 \,, \quad f^2 \,\mathrm{sh}^2 \,\alpha_0 = b^2 \,. \tag{10}$$

Since $\mathrm{ch}^2 \,\alpha_0 - \mathrm{sh}^2 \,\alpha_0 = 1$, the relation (8) gives:

$$\mathrm{ch}\, \alpha_0 = a/f \;\;, \quad \mathrm{sh}\, \alpha_0 = b/f \;\;, \tag{11}$$

or, alternatively,

$$e^{\alpha_0} = (a+b)/f \;\;, \quad \alpha_0 = \ln \left[(a+b)/f \right] \;. \tag{12}$$

The geometric flattening can be characterized also by the parameter $q_s = (a-b)/a$. In this manner we link the dimensions of our ellipsoid and transformation relations (7). We can also express the "source potential" V_0 given by (4) in the ellipsoidal coordinates as:

$$V_0\,(\alpha,\beta) = -\pi G \rho_0 f^2 \,\mathrm{ch}^2 \,\alpha \,\sin^2 \beta = -\frac{2}{3}\,\pi G \rho_0 f^2 \,\mathrm{ch}^2 \,\alpha [1 - P_2\,(\cos\beta)] \;, \tag{13}$$

where $P_2\,(\cos\beta) = \frac{1}{2}\left(3\cos^2\beta - 1\right)$ is the Legendre polynomial of degree two and argument $\cos\beta$. We can see that this "source potential" inside the ellipsoid is independent of the azimuthal angle φ. Since the form of the ellipsoid is azimuthally independent too, we use the azimuthally independent solutions of the Laplace equation:

$$\nabla^2 U(\alpha,\beta) = 0, \tag{14}$$

which has the form quoted in e.g. *Morse and Feshbach (1953)*, *Lebedev (1963)*:

$$\frac{1}{\mathrm{ch}\,\alpha}\frac{\partial}{\partial\alpha}\left(\mathrm{ch}\,\alpha\frac{\partial U}{\partial\alpha}\right) + \frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial U}{\partial\beta}\right) = 0. \tag{15}$$

Its particular solution is given by the functions:

$$U_n\,(\alpha,\beta) = \left\{ \begin{array}{c} P_n\,(\mathrm{i}\,\mathrm{sh}\,\alpha) \\ Q_n\,(\mathrm{i}\,\mathrm{sh}\,\alpha) \end{array} \right\} P_n\,(\cos\beta)\,, \tag{16}$$

where $P_n\,(\cos\beta)$ are the well known Legendre polynomials of degree n, argument $\cos\beta$. Specifically, we know that

$$P_0\,(x) = 1\,, \quad P_1\,(x) = x\,, \quad P_2\,(x) = \tfrac{1}{2}\left(3x^2 - 1\right)\,. \tag{17}$$

The functions $P_n\,(i\,\mathrm{sh}\,\alpha)$, $Q_n\,(i\,\mathrm{sh}\,\alpha)$ are less familiar spherical functions of the purely imaginary argument $(\mathrm{i}\,t)$, $t = \mathrm{sh}\,\alpha \in \langle 0, +\infty)$. The Legendre functions $P_n\,(\mathrm{i}\,t)$ are simply (*Smythe, 1968*; *Morse and Feshbach, 1953*):

$$P_0\,(\mathrm{i}\,t) = 1\,, \quad P_1\,(\mathrm{i}\,t) = \mathrm{i}\,t\,, \quad P_2\,(\mathrm{i}\,t) = -\tfrac{1}{2}\left(3t^2+1\right)\,. \tag{18}$$

Spherical functions of the second kind, $Q_n\,(\mathrm{i}\,t)$ have more complicated forms:

$$\begin{aligned} &Q_0\,(\mathrm{i}\,t) = -\,\mathrm{i}\ \mathrm{arctg}\,(t^{-1})\ , \quad Q_1\,(\mathrm{i}\,t) = t\ \mathrm{arctg}\,(t^{-1}) - 1\,, \\ &Q_2\,(\mathrm{i}\,t) = \mathrm{i}\left[\tfrac{1}{2}\,(3t^2+1)\,\mathrm{arctg}\,(t^{-1}) - \tfrac{3}{2}t\right]. \end{aligned} \tag{19}$$

A more general expression for $Q_n\,(\mathrm{i}\,t)$, valid for $t > 1$, takes the form (*Smythe, 1968*):

$$Q_n\,(\mathrm{i}\,t) = (-\,\mathrm{i})^{n+1}\,2^n \sum_{k=0}^{\infty} \frac{(-1)^k (n+2k)!(n+k)!}{k!(2n+2k+1)!t^{n+2k+1}}\,. \tag{20}$$

Analysis of functional properties of $Q_n\,(\mathrm{i}\ \mathrm{sh}\,\alpha)$ shows that these functions cannot occur in the interior potential $U_T^*\,(\alpha,\beta)$ since they would generate singular value of $grad\,U_T^*$ on the equatorial region $\beta = \pi/2$, $\alpha = 0$. This property is discussed in more detail by *Lebedev (1963)*. Functions $P_n\,(\mathrm{i}\ \mathrm{sh}\,\alpha)$ are singular as $\alpha \to +\infty$, i.e. far from the spheroid, so they cannot occur in the outer potential $U_1\,(\alpha,\beta)$. Then the potential $U_1\,(\alpha,\beta)$ is given by the sum of functions $Q_n\,(\mathrm{i}\ \mathrm{sh}\,\alpha)\,P_n\,(\cos\beta)$. The selection is straightforward, since the formula (13) shows that only the terms involving $P_0\,(\cos\beta) = 1$ and $P_2(\cos\beta)$ are non-zero. In potentials $U_T^*\,(\alpha,\beta)$, $U_1\,(\alpha,\beta)$ we have the terms of degrees $n = 0$, 2 only. The potential inside the ellipsoid is $U_T = V_0 + U_T^*$, so that

$$U_T\,(\alpha,\beta) = -w_0\,\mathrm{ch}^2\,\alpha[1 - P_2\,(\cos\beta)] + w_0E_0 + w_0E_2P_2\,(i\,\mathrm{sh}\,\alpha)\,P_2\,(\cos\beta), \tag{21}$$

where $w_0 = \frac{2}{3}\pi G\rho_0 f^2$. Since this potential is formed by the real functions of (α,β), as $P_2\,(\mathrm{i}\,\mathrm{sh}\,\alpha) = -\frac{1}{2}\left(3\,\mathrm{sh}^2\,\alpha + 1\right)$, we suppose that the outer potential $U_1\,(\alpha,\beta)$ is a real function as well,

$$U_1\,(\alpha,\beta) = w_0[D_0q_0\,(\mathrm{sh}\,\alpha) + D_2q_2\,(\mathrm{sh}\,\alpha)\,P_2\,(\cos\beta)], \tag{22}$$

where

$$q_0(t) = \mathrm{i}\, Q_0(\mathrm{i}\, t) = \operatorname{arctg}(1/t),\ \ q_2(t) = -\,\mathrm{i}\, Q_2(\mathrm{i}\, t) = \\ = \frac{1}{2}\left[(3t^2+1)\operatorname{arctg}(1/t) - 3t\right] \tag{23}$$

are real functions and $t = \operatorname{sh}\alpha$.

The multiplier constants E_0, E_2, D_0, D_2 can be determined from the boundary conditions on the surface $\alpha = \alpha_0$ of the ellipsoid. There must be continuity of potential and its normal derivative, with respect to α. At $\alpha = \alpha_0$ we have:

$$[\,U_T\,]_{\alpha=\alpha_0} = [\,U_1\,]_{\alpha=\alpha_0}\,, \tag{24}$$

$$\left[\frac{\partial U_T}{\partial \alpha}\right]_{\alpha=\alpha_0} = \left[\frac{\partial U_1}{\partial \alpha}\right]_{\alpha=\alpha_0}. \tag{25}$$

We will apply these continuity conditions separately for $P_0\,(\cos\beta) = 1$ and $P_2\,(\cos\beta)$. The terms of the order $n = 0$ produce the two equations:

$$\begin{aligned} -\operatorname{ch}^2\alpha_0 + E_0 &= D_0 q_0\,(\operatorname{sh}\alpha_0)\ , \\ -2\operatorname{ch}\alpha_0\operatorname{sh}\alpha_0 &= D_0 q_0'\,(\operatorname{sh}\alpha_0)\operatorname{ch}\alpha_0\,. \end{aligned} \tag{26}$$

The second equation yields:

$$D_0 = -\frac{2\operatorname{sh}\alpha_0}{q_0'\,(\operatorname{sh}\alpha_0)} = 2\left(\frac{b}{f}\right)\left(\frac{a}{f}\right)^2. \tag{27}$$

Here we have used the relations which follow from (22) and (18), namely $q_0\,(t) = \operatorname{arctg}\,(1/t)$, $q_0'\,(t) = -1/(1+t^2)$, and the following properties from (11)

$$t_0 = \operatorname{sh}\alpha_0 = \frac{b}{f}\,, \quad t_0^2 + 1 = \operatorname{ch}^2\alpha_0 = \left(\frac{a}{f}\right)^2. \tag{28}$$

After simple manipulations the constant E_0 in the interior potential takes the form after simple modifications:

$$E_0 = \left(\frac{a}{f}\right)^2\left[1 + 2\left(\frac{b}{f}\right)\operatorname{arctg}\left(\frac{f}{b}\right)\right]. \tag{29}$$

The continuity of terms proportional to $P_2\,(\cos\beta)$ in the boundary conditions (24) and (25) yield the two equations for E_2, D_2:

$$\begin{aligned} E_2 p_2 (\operatorname{sh}\alpha_0) - D_2 q_2 (\operatorname{sh}\alpha_0) &= -\operatorname{ch}^2\alpha_0 \,, \\ E_2 p_2' (\operatorname{sh}\alpha_0) - D_2 q_2' (\operatorname{sh}\alpha_0) &= -2\operatorname{sh}\alpha_0 \,. \end{aligned} \tag{30}$$

Here we have used that

$$P_2 (\mathrm{i}\, t) = -\tfrac{1}{2}\left(3t^2+1\right) \equiv p_2 (t) \ , \quad p_2' (t) = -3t \,, \tag{31}$$

where $p_2(t)$ is a real function of t. The determinant of this system is

$$\begin{aligned} X_2 &= -\left[p_2 (\operatorname{sh}\alpha_0)\, q_2' (\operatorname{sh}\alpha_0) - p_2' (\operatorname{sh}\alpha_0)\, q_2 (\operatorname{sh}\alpha_0)\right] = \\ &= -\frac{1}{1+\operatorname{sh}^2\alpha_0} = -\frac{1}{\operatorname{ch}^2\alpha_0} \,. \end{aligned} \tag{32}$$

Here we have used the relation for the Wronskian of functions $P_n(\mathrm{i}\operatorname{sh}\alpha)$, $Q_n(\mathrm{i}\operatorname{sh}\alpha)$ which, according to *Lebedev (1963)*, is:

$$P_n (\mathrm{i}\, t)\, Q_n' (\mathrm{i}\, t) - P_n' (\mathrm{i}\, t)\, Q_n (\mathrm{i}\, t) = -\frac{1}{t^2+1}, \tag{33}$$

including transformation relations $P_2(\mathrm{i}\, t) = p_2(t)$, $Q_2(\mathrm{i}\, t) = \mathrm{i}\; q_2(t)$ and their derivatives with respect to t. Then we obtain:

$$E_2 = -\operatorname{ch}^2\alpha_0\left[\operatorname{ch}^2\alpha_0\, q_2' (\operatorname{sh}\alpha_0) - 2\operatorname{sh}\alpha_0\, q_2 (\operatorname{sh}\alpha_0)\right], \tag{34}$$

$$\begin{aligned} D_2 &= \operatorname{ch}^2\alpha_0\left[2\operatorname{sh}\alpha_0\, p_2 (\operatorname{sh}\alpha_0) - \operatorname{ch}^2\alpha_0\, p_2' (\operatorname{sh}\alpha_0)\right] = \\ &= 2t_0\left(t_0^2+1\right) = 2\left(\frac{b}{f}\right)\left(\frac{a}{f}\right)^2 . \end{aligned} \tag{35}$$

Here we have again used the relations (11) for t_0. Now we can write the explicit formula for the external potential $U_1(\alpha, \beta)$:

$$U_1 (\alpha,\beta) = w_0 [D_0 q_0 (\operatorname{sh}\alpha) + D_2 q_2 (\operatorname{sh}\alpha)\, P_2 (\cos\beta)]. \tag{36}$$

We can easily find the following relations using $w_0 = \frac{2}{3}\pi G\rho_0 f^2$:

$$w_0 D_0 = \frac{GM}{f} = \frac{GM}{\sqrt{a^2-b^2}}, \tag{37}$$

where $M = \frac{4}{3}\pi a^2 b\rho_0$ is the total mass of the ellipsoid. Also,

$$w_0 D_2 = GM/f. \tag{38}$$

Finally we have

$$U_1(\alpha, \beta) = \frac{GM}{f}[\operatorname{arctg}(1/\operatorname{sh}\alpha) + q_2(\operatorname{sh}\alpha)\, P_2(\cos\beta)]. \tag{39}$$

This formula coincides with the classical one (e.g. *Grushinskyj, 1963*), but his derivations are more complicated than ours.

3. Calculation of gravity components and numerical examples

After solving the boundary value problem for potentials $U_T(\alpha, \beta)$ and $U_1(\alpha, \beta)$, we can write formulae for components of the gravity acceleration: $\boldsymbol{g} = \operatorname{grad} U(\alpha, \beta)$. These are as follows:

$$g_\alpha = \frac{1}{h_\alpha}\frac{\partial U}{\partial \alpha}, \quad g_\beta = \frac{1}{h_\beta}\frac{\partial U}{\partial \beta}, \tag{40}$$

where the Lame's metric parameters are $h_\alpha = h_\beta = f\left(\operatorname{ch}^2\alpha - \sin^2\beta\right)^{1/2}$. Using formula (21) for $U_T(\alpha, \beta)$, the gravity field inside the spheroid takes the form:

$$\begin{aligned} g_{T\alpha} &= -w_0 h_\alpha^{-1} 3\operatorname{ch}\alpha\operatorname{sh}\alpha\left[\sin^2\beta + E_2 P_2(\cos\beta)\right], \\ g_{T\beta} &= -w_0 h_\beta^{-1} 3\cos\beta\sin\beta\left[\operatorname{ch}^2\alpha + \tfrac{1}{2}E_2(3\operatorname{sh}^2\alpha + 1)\right], \end{aligned} \tag{41}$$

where we have used relations (31). Similar treatment can be applied to the outer potential $U_1(\alpha, \beta)$ using (39), which gives

$$\begin{aligned} g_{1\alpha} &= (GM/f)h_\alpha^{-1}\left[-1/\operatorname{ch}\alpha + q_2'(\operatorname{sh}\alpha)\operatorname{ch}\alpha P_2(\cos\beta)\right], \\ g_{1\beta} &= -(GM/f)h_\beta^{-1} q_2(\operatorname{sh}\alpha) 3\sin\beta\cos\beta. \end{aligned} \tag{42}$$

These curvilinear components can be transformed into Cartesian components g_x, g_z in the plane (x, z) i.e. $y = 0$ which is sufficient for for determination of the whole gravity pattern due to oblate ellipsoid. Using modified formulae e.g. from *Madelung (1957)*, we obtain relations:

$$\begin{aligned} g_x &= [g_\alpha\operatorname{sh}\alpha\sin\beta + g_\beta\operatorname{ch}\alpha\cos\beta]\,(\operatorname{ch}^2\alpha - \sin^2\beta)^{-1/2}\cdot\operatorname{sign}(x), \\ g_z &= [g_\alpha\operatorname{ch}\alpha\cos\beta - g_\beta\operatorname{sh}\alpha\sin\beta]\,(\operatorname{ch}^2\alpha - \sin^2\beta)^{-1/2}, \end{aligned} \tag{43}$$

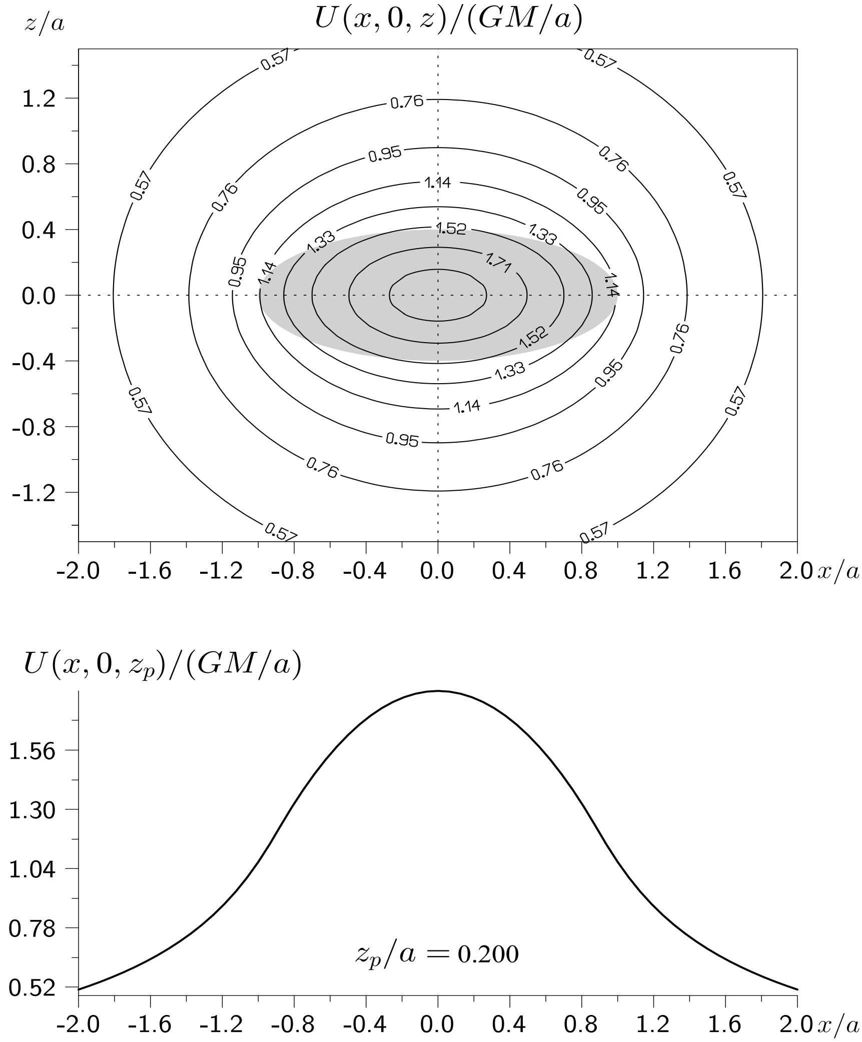
where sign(x) represents the value of $\cos\varphi$ for $\varphi = 0, \pi$. For the practical purposes we will calculate the gravity field over the regular network of points (x, z) in the plane $y = 0$. We need to assign the ellipsoidal coordinates (α, β) to each (x, z). We can calculate them by using the property that the coordinate line $\alpha =$ const is an ellipse described by Eq. (9) in the x, z plane; their foci lie on the focal circle $r = f$ in the plane $z = 0$, major semiaxis length is $f \operatorname{ch}\alpha$ and the minor semiaxis is of the length $f \operatorname{sh}\alpha$. For every point (x, z) of this ellipse the sum of distances from the first and second foci is equal to the doubled value of the major semiaxis length, $2f \operatorname{ch}\alpha$. Then:

$$\left[(r-f)^2 + z^2\right]^{1/2} + \left[(r+f)^2 + z^2\right]^{1/2} = 2f \operatorname{ch}\alpha, \tag{44}$$

where $r = (x^2 + y^2)^{1/2} = |x|$ and $f = (a^2 - b^2)^{1/2}$ is the focal constant given by the dimensions of the basic ellipse $\alpha = \alpha_0$. If we determine $\operatorname{ch}\alpha$, we have also $\operatorname{sh}\alpha = (\operatorname{ch}^2\alpha - 1)^{1/2}$ and $\mathrm{e}^\alpha = \operatorname{ch}\alpha + \operatorname{sh}\alpha$. The angle β, reckoned from the semiaxis $z \geq 0$, can be determined from equations

$$\cos\beta = z/(f \operatorname{sh}\alpha), \quad \sin\beta = \sqrt{x^2 + y^2}/(f \operatorname{ch}\alpha). \tag{45}$$

The formula for $\sin\beta$ is valid also inside the focal circle $z = 0$, $r < f$. Now we have all the formulae necessary for numerical calculations. We present the results for two cases of ellipsoid, corresponding to ratios $b/a = 0.4, 0.1$. In Fig. 2a–c we can see the isolines of potential and components g_α, g_β for the case $b/a = 0.4$. The values of potential are scaled by the value GM/a, while the gravity components are scaled by the value GM/a^2 and multiplied by the factor 10, since the body is rather small ($a = 10\,\mathrm{m}$) and its density $\rho = 1000\,\mathrm{kg\,m^{-3}}$ (like water). In Figs. 2a–c a profile curve for the level $z_p/a = 0.2$ is also plotted. In Fig. 2a we can see that the equipotential surfaces are almost elliptic, but the surface of the spheroid $\alpha = \alpha_0$ is not equipotential, simply because of the presence of the terms E_2, D_2 in potentials. The pattern of isolines g_α is presented in Fig. 2b. We can see that all values g_α are negative, indicating that the direction of the gravity vector is downwards to the interior of the ellipsoid. The continuity of g_α on the surface of the ellipsoid $\alpha = \alpha_0$ is preserved (the small ripples of the isoline curves there are due to an artefact of the plotting program). We can also see that $|g_\alpha|$ attains maximum on the boundary $\alpha = \alpha_0$, being most visible on the profile curve for $z_p/a = 0.2$. The pattern



$a, b, f = 10.000, 4.000, 9.165\,\mathrm{m}$
$prof : z_p = 2.00\,\mathrm{m}$
$\alpha_0, q_s = 0.4236, 0.6000$
$sh(\alpha_0) = 0.4364$, $GM/a = 0.11183\text{E-}04\,\mathrm{m}^2/\mathrm{s}^2$

Fig. 2a. Isolines of the gravity potential in the plane $y_c = 0$ inside and outside the oblate ellipsoid with $b/a = 0.4$, i.e. $q_s = (a - b)/a = 0.6$. The bottom graph presents the profile curve for the plane $z_p/a = 0.2$.

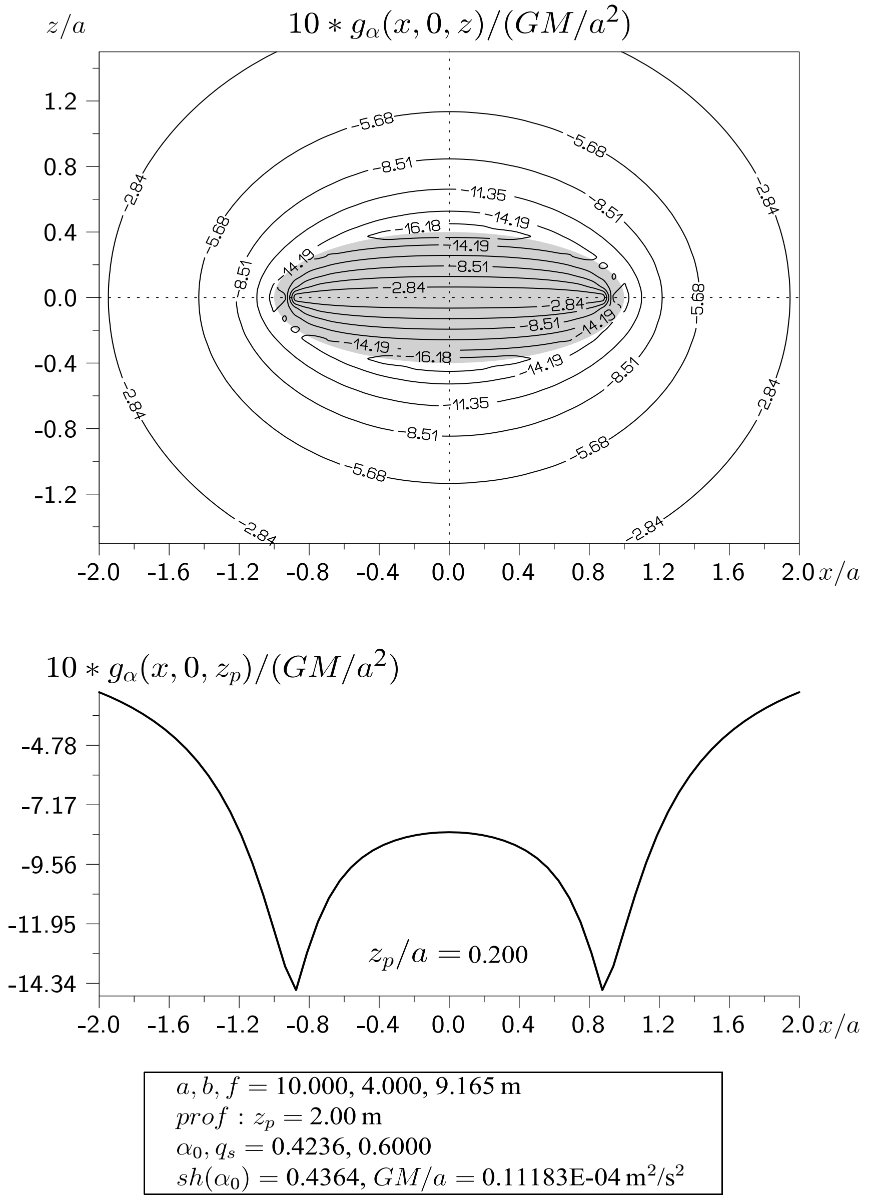
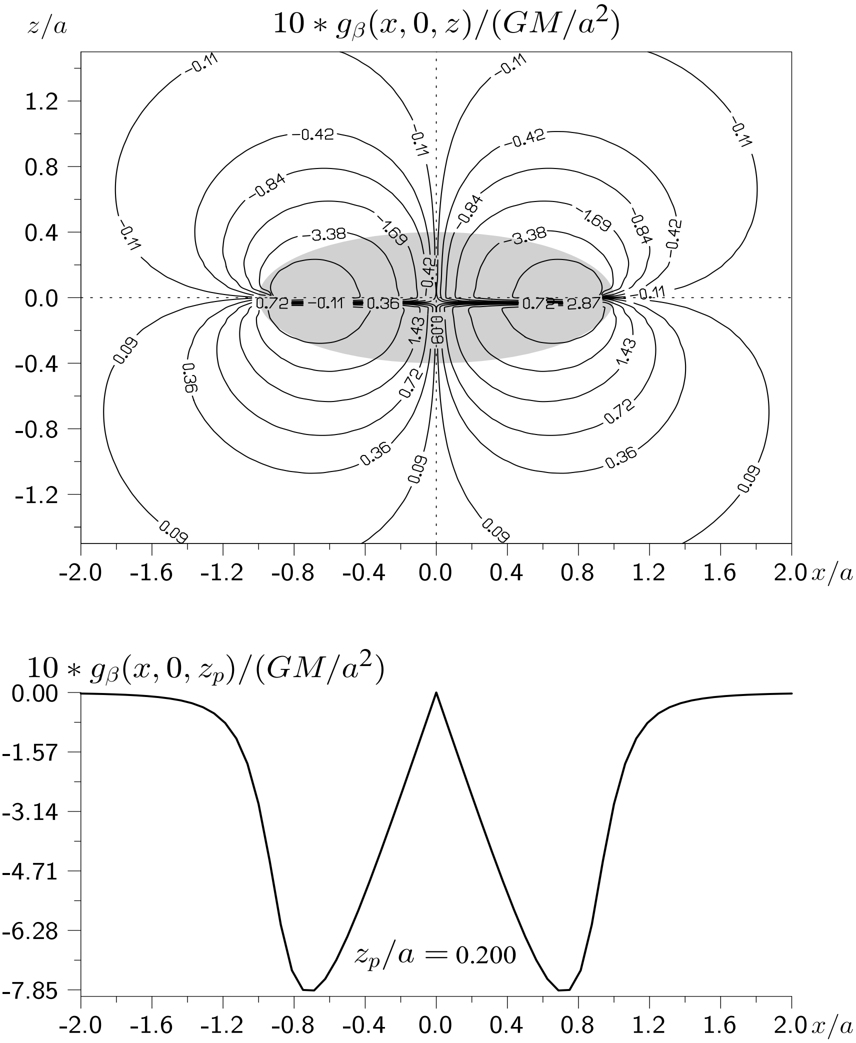


Fig. 2b. The same as in Fig. 2a, but for the normalized component g_α of the gravity acceleration.



$a, b, f = 10.000, 4.000, 9.165\,\mathrm{m}$
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$\alpha_0, q_s = 0.4236, 0.6000$
$sh(\alpha_0) = 0.4364$, $GM/a = 0.11183\text{E-}04\,\mathrm{m^2/s^2}$

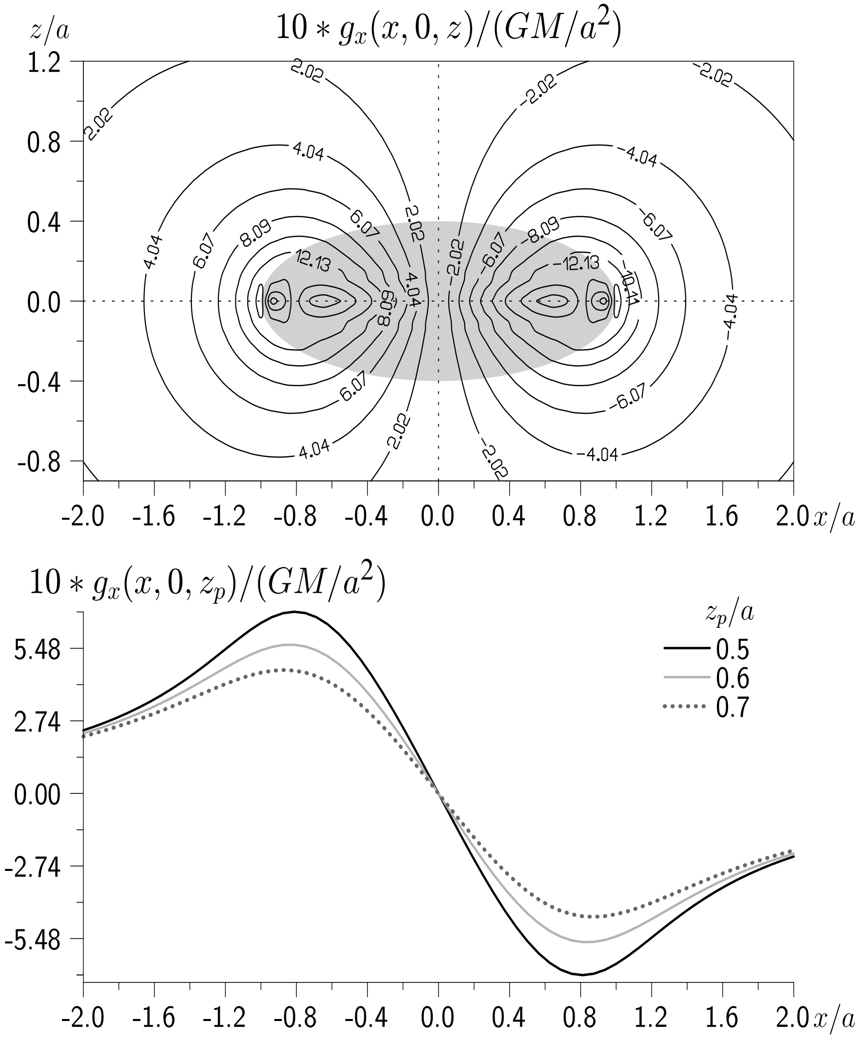
Fig. 2c. The same as in Fig. 2a, but for the normalized component g_β of the gravity acceleration.

of the component g_β is more complicated, because it is proportional to the factor $\cos\beta\sin\beta$, as follows from formulae (41), (42). Rather surprising were the negative values of g_β in the halfplane $z > 0$, which indicates that the vector $\boldsymbol{g} \equiv (g_\alpha, g_\beta)$ is inclined to the polar semiaxis $z \geq 0$. From the physical point of view, however, it confirms that the gravity vector points nearly towards the gravity centre of the body. This is confirmed also by the positive values of g_β in the halfplane $z < 0$. It is also clear that g_β values are zero on the polar axis z $(\beta = 0, \pi)$ and in the equatorial plane $\beta = \pi/2$. For the purposes of applied geophysics it is more useful to know the Cartesian components of the gravity of the oblate ellipsoid. For this reason we present in Figs. 3a–b the g_x, g_z isoline graphs for the same ellipsoid as used in Figs. 2a–c. In Fig. 3a we can see that g_x is positive for the $x < 0$ halfplane, while for the $x > 0$ halfplane it is negative. This is in accord with qualitative expectation that the gravity vector points into the attracting body. This property is also clear from the profile curves for 3 levels $z_p/a = 0.5, 0.6, 0.7$, which are above the spheroid. The g_z isoline graph on Fig. 3b shows that this component is negative for the upper halfplane $(z \geq 0)$, while for the halfplane $z < 0$ the values are positive. This also matches with the general feature of the gravity vector – it points inside the attracting body. In the equatorial plane $z = 0$ we have $g_z = 0$, while the gravity vector is given by the maximal values of g_x. For the purposes of applied gravimetry we have plotted in bottom parts of Figs. 3a–b the profile curves of g_x and Δg at three planes above the ellipsoid $z_p/a = 0.5, 0.6, 0.7$. Note that in Fig. 3b there is plotted $\Delta g(x, 0, z_p) = -g_z(x, 0, z_p)$ because in our model we have the z axis oriented upward in contrast to traditional downward orientation in the applied gravimetry.

We also calculated gravity field for the relatively thin oblate ellipsoid $(b \ll a)$ which approximates the circular disc of radius a. The thin disc of radius a can be considered as a very thin oblate spheroid with $b \ll a$ and hence $f = (a^2 - b^2)^{1/2} \doteq a$. The gravity potential due to this disc is, according to (39),

$$U_1(\alpha, \beta) = \frac{GM}{a}\left\{\mathrm{arctg}(1/sh\alpha) + q_2(\mathrm{sh}\,\alpha)P_2(\cos\beta)\right\}. \tag{46}$$

The transformation relations (44) and (45) are also slightly modified by using $f \doteq a$. For the purpose of illustration we set $a = 10\,\mathrm{m}$, $b = 1\,\mathrm{m}$ and the value $GM/a = 0.27957 \times 10^{-5}\,\mathrm{m}^2\,\mathrm{s}^{-2}$. The results are presented in



$a, b, f = 10.000, 4.000, 9.165\,\mathrm{m}$
$\alpha_0, q_s = 0.4236, 0.6000$
$sh(\alpha_0) = 0.4364, GM/a = 0.11183\text{E-}04\,\mathrm{m}^2/\mathrm{s}^2$

Fig. 3a. Isolines of the horizontal gravity component g_x for the potential presented in Figure 2a. There are also plotted three profile curves for planes $z_p/a = 0.5, 0.6, 0.7$.

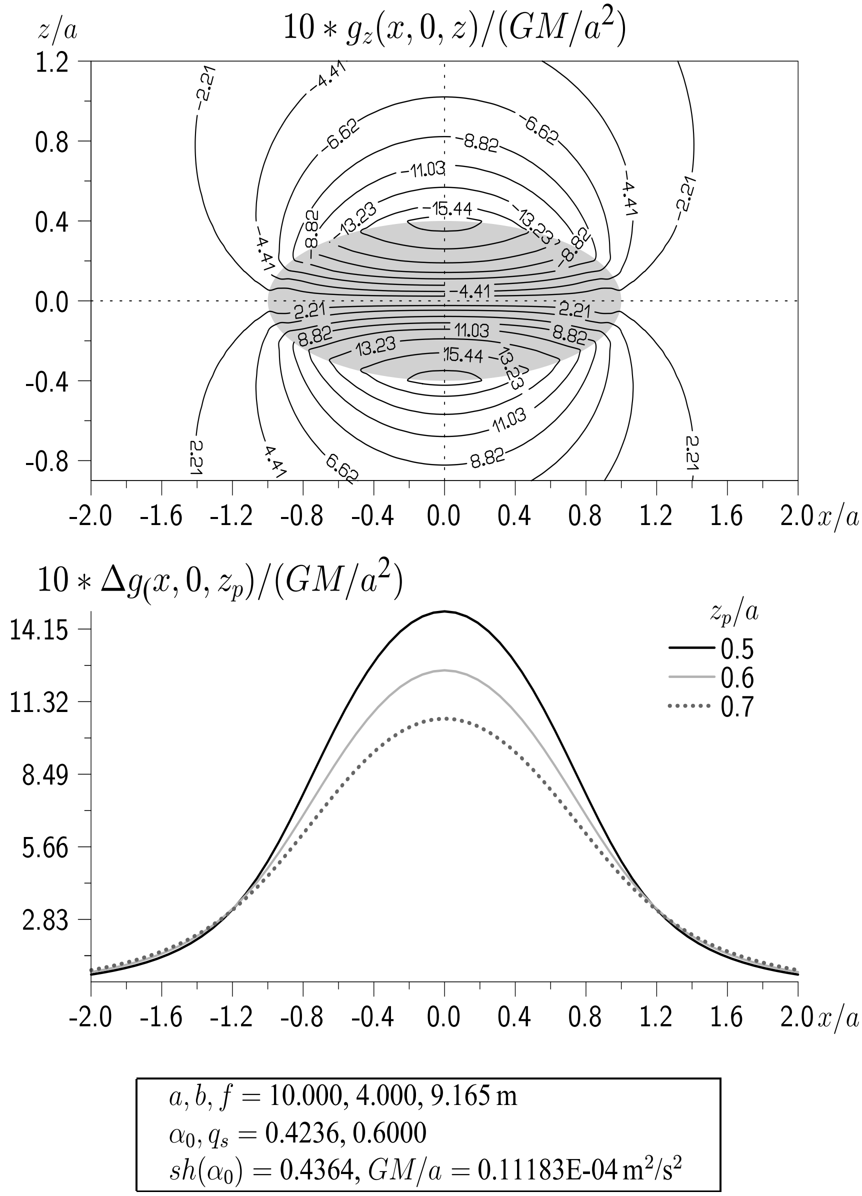


Fig. 3b. Isolines of the vertical gravity component g_z for the potential presented in Figure 2a. There are also plotted three profile curves of gravity anomaly: $\Delta g = -g_z$ at planes $z_p/a = 0.5, 0.6, 0.7$.

Figs. 4a–c for the upper halfplane only, $z > 0$. The equipotential isolines are similar to those given in Fig. 2a. The Cartesian components of gravity $g_x, \Delta g$ are also similar to those in Figs. 3a,b, but the extremal values are different. It is interesting that the boundary of the disc edge $x/a = \pm 1$ coincides with the maximum for the horizontal component. The vertical component Δg grows towards the disc most rapidly along the z axis. There is a well-known formula for Δg along the z-axis, $\Delta g(0,z) = 2GMa^{-2}[1 - z(a^2+z^2)^{-1/2}]$, $(x = 0)$, which is confirmed by the values of $10 \times \Delta g$ for the profiles $z_p/a = 0.5, 0.6, 0.7$ at $x = 0$ in the Fig. 4c.

It is natural to ask about the applicability of formula (39) to the case of sphere, when $b \to a$ and $f \to 0$. For this purpose we use the transform formulae (7) which give the expression for the distance R from the centre of sphere:

$$R^2 = x^2 + y^2 + z^2 = f^2(\operatorname{sh}^2\alpha + \sin^2\beta). \tag{47}$$

It is clear that the potential of the sphere must be independent of β, so we use R for $\beta = 0$:

$$R = f \operatorname{sh}\alpha, \quad \operatorname{sh}\alpha = R/f, \tag{48}$$

which means that we have to calculate the limit of (39) for $f \to 0$. It can be easily shown that

$$\lim_{f\to 0} f^{-1} q_2(R/f) = 0, \quad \lim_{f\to 0} f^{-1} \operatorname{arctg}(f/R) = R^{-1}, \tag{49}$$

so we have for the potential of the sphere the classical formula

$$\lim_{f\to 0} U_1(\alpha, 0) = U_1(R) = GM/R. \tag{50}$$

Finally, we can state that our formulae are valid for this classical limit case of oblate spheroid.

4. Conclusion

In this paper we have applied both analytically and numerically the oblate ellipsoidal functions (16) to calculate the gravity field due to oblate ellipsoid

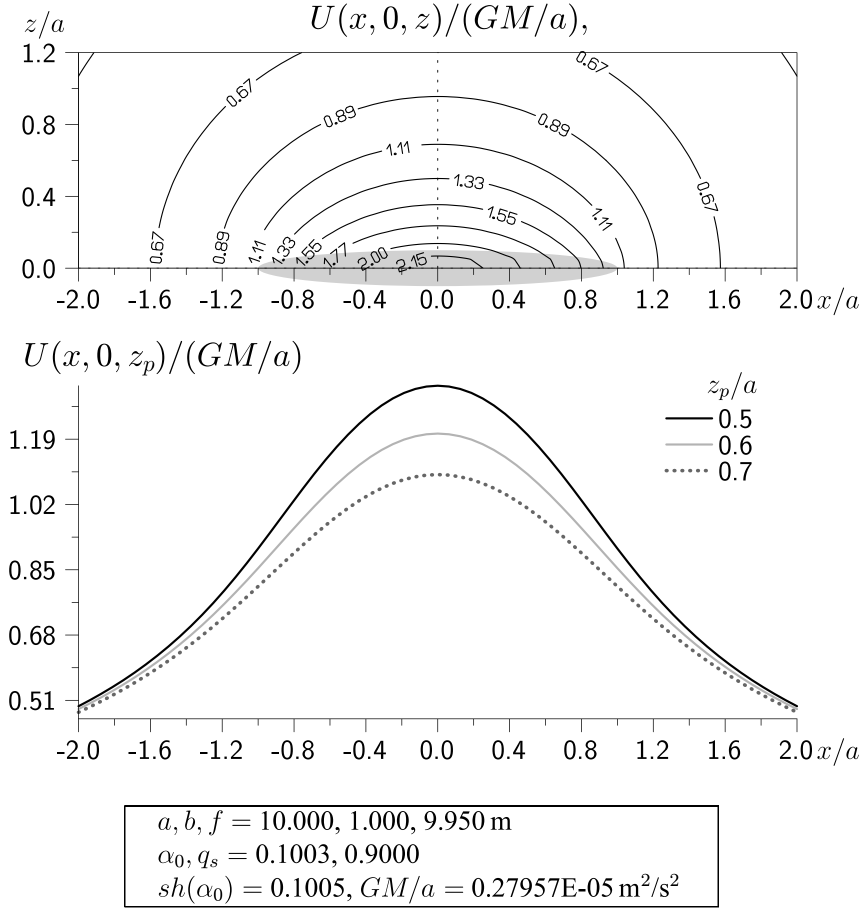


Fig. 4a. Isolines of the gravity potential due to very oblate ellipsoid with $b/a = 0.1$ (circular disc) for the upper halfplane $z > 0$, $y = 0$. The scaling value $GM/a = 0.27957 \times 10^{-5}\,\mathrm{m}^2/s^2$. The three profile curves for planes $z_p/a = 0.5, 0.6, 0.7$ are also plotted.

of uniform density for the general ratio $b/a \in (0, 1)$, which covers also the limit cases of disc ($b/a \to 0$) or sphere ($b/a \to 1$). The advantage of our treatment is that the potentials outside or inside the body are expressed by the two terms in variables (α, β), with a simple transformation to the Cartesian ones (x, z). Similar treatment can also be used for the calculation of the gravity potential for the oblate ellipsoid composed by confocal ellipsoidal layers of different density. We note that the oblate spheroidal bodies including disc are of interest in the astrophysics e.g. *Syer (1995); Cheng et al. (2007)*.

Finally we note that our formulae can be easily extended also to the rotating spheroid like the Earth. For this case we must include in the potential $U_T(\alpha,\beta)$ also the potential of the centrifugal acceleration $U_c(\alpha,\beta) = \frac{1}{2}\Omega^2(x^2+y^2) = \frac{1}{2}f^2\Omega^2\,\mathrm{ch}^2\,\alpha\sin^2\beta$, where Ω is the angular frequency of the rotation around the z axis. Since in our treatment we have considered el-

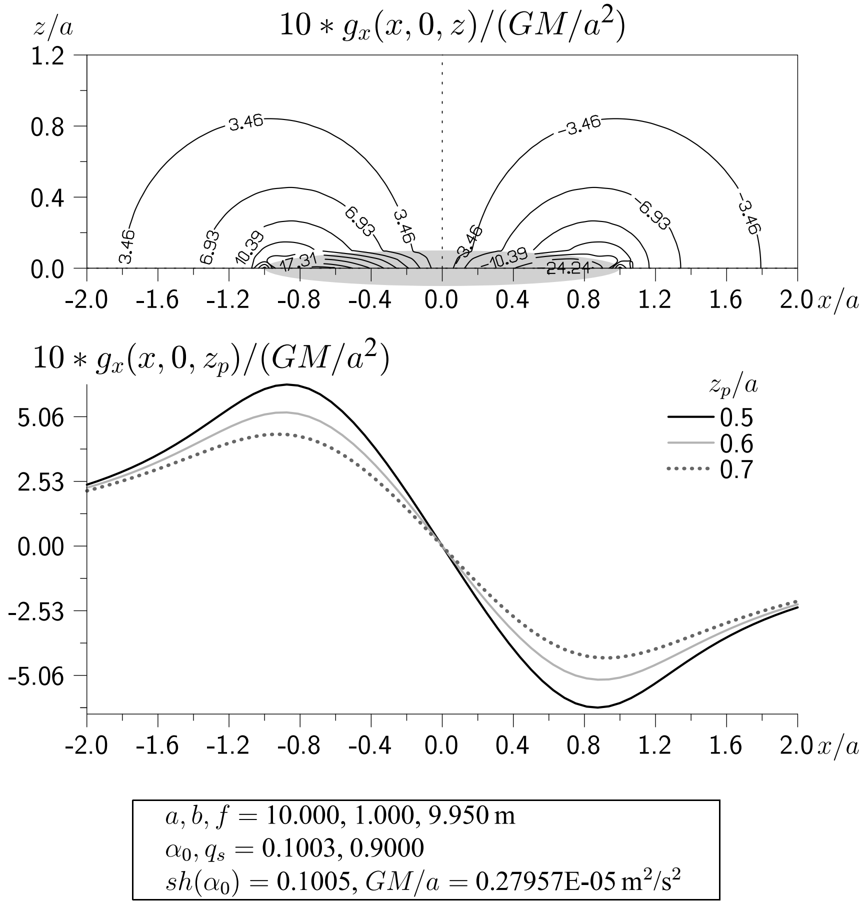
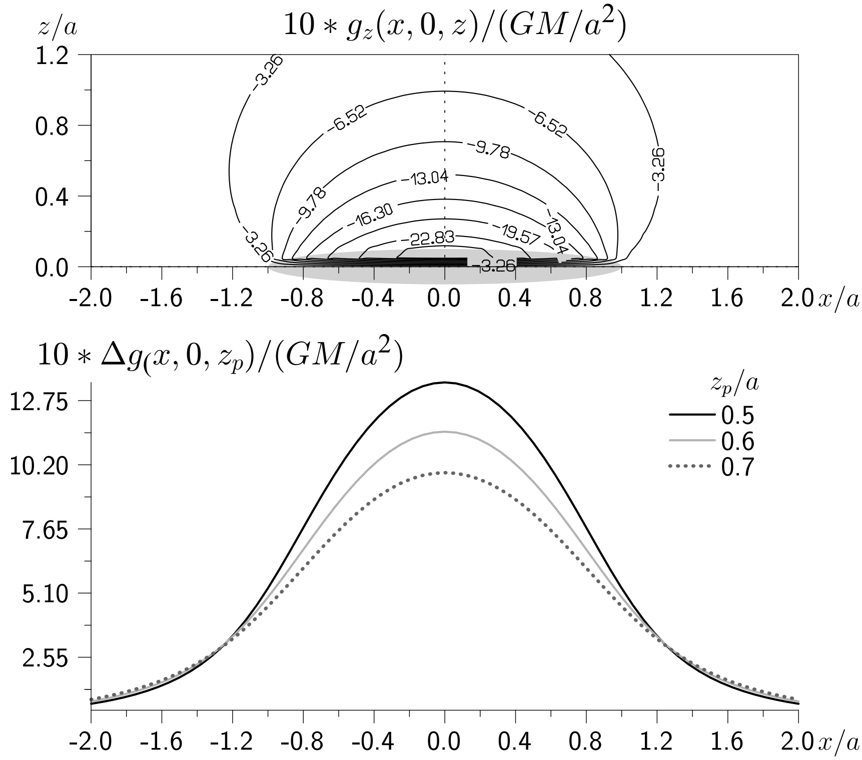


Fig. 4b. Isolines of the horizontal gravity component g_x for the potential presented in Fig. 4a. The three profile curves for planes $z_p/a = 0.5, 0.6, 0.7$ are also plotted.



$a, b, f = 10.000, 1.000, 9.950\,\mathrm{m}$
$\alpha_0, q_s = 0.1003, 0.9000$
$sh(\alpha_0) = 0.1005, GM/a = 0.27957\mathrm{E}\text{-}05\,\mathrm{m}^2/\mathrm{s}^2$

Fig. 4c. Isolines of the vertical gravity component g_z for the potential presented in Fig. 4a. There are also plotted three profile curves of $\Delta g = -g_z$ at planes $z_p/a = 0.5, 0.6, 0.7$.

lipsoids of dimensions suitable for the exploration gravimetry, we do not discuss this rotational case. This is interesting e.g. for the case of Earth with low flattening $b/a \to 1$, $a = R_e$. It is clear that the acceleration due to rotation will decrease both g_α and g_β.

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References

Arfken G., 1966: Mathematical methods for physicists, Academic Press London, 1966.

Cheng H., Song G., Huang Ch., 2007: The internal and external metrics of a rotating ellipsoid under Post-Newtonian approximation. Chinese Astronomy and Astrophysics, 31, 192–204.

Duboshin G. N., 1961: The gravity theory. GIFML, Moscow (in Russian).

Grafarend E. W., Klapp M., Martinec Z., 2010: Space time modeling of the Earth's gravity field by ellipsoidal harmonics. Handbook of Geomathematics, Springer Verlag, Heidelberg.

Grushinskyj N. P., 1963: Theory of the Earth's figure. GIFML, Moscow (in Russian).

Heiskanen W. A., Moritz H., 1967: Physical Geodesy. Freeman, San Francisco.

Hobson E. W., 1931: The theory of spherical and ellipsoidal harmonics. Cambridge Univ. Press.

Kellog O. D., 1929: Foundations of potential theory. Dover Publ., New York.

Lebedev N. N., 1963: Special functions and their applications. Fizmatgiz, Moscow (in Russian).

MacMillan W. D., 1958: Theory of potential. Dover Publ., New York.

Madelung E., 1957: Die Mathematischen Hilfsmittel des Physikers, Springer-Verlag, Berlin.

Morse P. M., Feshbach H., 1973: Methods of theoretical physics. McGraw-Hill Comp., New York.

Muratov R. Z., 1976: The ellipsoid's potentials. Atomizdat, Moscow (in Russian).

Pick M., Pícha J., Vyskočil V., 1973: Theory of the Earth's gravity field. Elsevier, Amsterdam.

Smythe W. R., 1968: Static and dynamic electricity. McGraw-Hill, New York.

Syer D., 1995: Harmonic function expansion of nearly oblate systems. Mon. Not. R. Astron. Soc., 276, 1009–1016.

Wang W. X., 1988: The potential for a homogeneous spheroid in a spheroidal coordinate system: I. At an exterior point. J. Phys. A. Math. Gen., 21, 4245–4250.

Wang W. X., 1989: The potential for a homogeneous spheroid in a spheroidal coordinate system: II. At an interior point. J. Phys. A. Math. Gen., 22, 1459–1462.