Uniform spectral representation of the Earth's inner density structures and their gravitational field

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Abstract: We introduce the generic expressions for computing the gravitational field (potential and its radial derivative) generated by an arbitrary density (contrast) layer with a variable depth and thickness having a laterally-distributed radial density variation. The information on the geometry and density distribution of a volumetric mass layer is described by means of spherical harmonics. These generic expressions can uniformly be applied to model all major known density structures within the Earth's interior using methods for a spherical harmonic analysis and synthesis of the gravitational field. This is demonstrated on specific examples given for various density models commonly adopted for the approximation of crust density structures.

Key words: density, Earth interior, forward modelling, gravity, spherical harmonics

1. Introduction

The methods for a spherical harmonic analysis and synthesis of gravity field have been utilised in the gravimetric forward modelling by a number of authors. For the literature overview of these methods we refer readers to *Tenzer et al. (2010b)*. Assuming a constant density distribution, *Sünkel (1968)* derived the expressions for computing the topographic and topographic-isostatic potentials by means of spherical height functions. *Pavlis and Rapp (1990)* developed the global isostatic gravitational model complete to degree

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360 (based on the Airy-Heiskanen isostatic hypothesis) by combining low-degree satellite-derived geopotential models with the harmonic coefficients of the topographic-isostatic potential (taking into consideration also the gravitational effect of the continental ice). More complex density distribution models were taken into consideration in order to improve the accuracy of the gravimetric forward modeling of the Earth's density structures including the atmosphere. A more refined form of spectral expressions which takes into account the lateral density distribution was presented by *Sjöberg (1998)* and others. A change of atmospheric density with elevation was assumed in computing the atmospheric gravitational effects, for instance, by *Sjöberg and Nahavandchi (2000)*. *Tenzer et al. (2011a)* facilitated a depth-depended seawater density model in computing the bathymetric stripping gravity corrections. *Tenzer et al. (2010a)* derived and applied expressions for computing the ice density contrast stripping corrections to gravity field. *Tenzer et al. (2011b)* introduced expressions for computing the gravitational field generated by the laterally varying and homogeneous mass density contrast layer with a variable depth and thickness in terms of spherical harmonics. These expressions allow the fast and effective gravimetric forward modelling of density structures within the Earth's crust based on currently available global crustal models. For a more realistic approximation of geological and other density structures and an improvement of the numerical efficiency more complex density models can be used which take into consideration both, the lateral and depth-dependent density variations. One example can be given by sedimentary basins where the density increases with depth due to compaction (cf. *Artemjev et al., 1994*). In this study we formulate the expressions for computing the gravitational field quantities generated by an arbitrary mass density (contrast) layer with a variable depth and thickness having a laterally distributed radial density variation. We demonstrate that these generic expressions can uniformly be applied in the gravimetric forward modelling of all major Earth's inner density structures. Theoretical principles are reviewed in Section 2. The generic expressions are introduced in Section 3. In Sections 4–6, we apply these expressions for computing the gravitational field of density structures which are approximated by volumetric layers with specifically defined density distribution models. Examples of applying these generic expressions in the forward modelling of the topographic and (bathymetric, ice, sediments and

consolidated crust components) stripping gravity corrections are given and discussed in Section 7. The summary and concluding remarks are given in Section 8. We note that the application of isostatic models (see e.g., *Pavlis and Rapp, 1990*) is out of the scope of this study.

2. Gravitational field of volumetric layer, density model

In the context of the gravimetric forward modelling of the Earth's inner density structures, we formally distinguish two cases representing the gravitational contributions of density masses distributed above and below the geoid surface. In spherical approximation, the gravitational potential V generated by an arbitrary volumetric mass density layer beneath the geoid surface with a variable depth and thickness computed at a position (r, Ω) is defined by the following spatial representation of Newton's volume integral

$$V(r, \Omega) = \mathrm{G} \iint\limits_{\Phi} \int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}-D_U(\Omega')} \rho(r', \Omega') \; \ell^{-1}(r, \psi, r') \; r'^2 \mathrm{d}r' \, \mathrm{d}\Omega'. \tag{1}$$

For the volumetric layer above the geoid surface, the potential V reads

$$V(r, \Omega) = \mathrm{G} \iint\limits_{\Phi} \int_{\mathrm{R}+H_L(\Omega')}^{\mathrm{R}+H_U(\Omega')} \rho(r', \Omega') \; \ell^{-1}(r, \psi, r') \; r'^2 \mathrm{d}r' \, \mathrm{d}\Omega', \tag{2}$$

where $\mathrm{G} = 6.674 \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$ is Newton's gravitational constant; $\mathrm{R} = 6371 \times 10^3\,\mathrm{m}$ is the Earth's mean radius (which approximates the geocentric radius of the geoid surface); D_U and D_L are the depths (reckoned relative to the sphere of radius R) of the upper and lower bounds of the volumetric layer (beneath the geoid surface), respectively; H_U and H_L are the corresponding heights for the volumetric layer above the geoid surface; ℓ is the Euclidean spatial distance between positions of the computation point (r, Ω) and the integration (running) point (r', Ω'), and ψ is the respective spherical distance; $\mathrm{d}\Omega' = \sin\phi' \, \mathrm{d}\phi' \, \mathrm{d}\lambda'$ is the infinitesimal surface element of the unit sphere; the full spatial angle is denoted as $\Phi = \{\Omega' = (\phi', \lambda') : \phi' \in [-\pi/2, \pi/2] \wedge \lambda' \in [0, 2\pi]\}$; and $\rho(r', \Omega')$ is the mass density function. The 3D position is defined in geocentric spherical coordinates (r, ϕ, λ), where r is the geocentric radius and $\Omega = (\phi, \lambda)$ denotes the geocentric direction with the geocentric spherical latitude ϕ and longitude λ. The density function $\rho(r', \Omega')$ within the volumetric mass layer in Eqs. (1) and (2) is approximated by the laterally distributed radial density variation model using the following polynomial function (for each lateral column)

$$\rho\left(r', \Omega'\right) = \rho\left(D_U, \Omega'\right) + \beta\left(\Omega'\right) \sum_{i=1}^{I} a_i\left(\Omega'\right) \left(\mathrm{R} - r'\right)^i \quad \left[\mathrm{R} - D_U\left(\Omega'\right) \geq r' \geq \mathrm{R} - D_L\left(\Omega'\right) : \Omega' \in \Phi\right], \tag{3}$$

and

$$\rho\left(r', \Omega'\right) = \rho\left(H_U, \Omega'\right) + \beta\left(\Omega'\right) \sum_{i=1}^{I} a_i\left(\Omega'\right) \left(\mathrm{R} - r'\right)^i \quad \left[\mathrm{R} + H_U\left(\Omega'\right) \geq r' \geq \mathrm{R} + H_L\left(\Omega'\right) : \Omega' \in \Phi\right], \tag{4}$$

where $\rho(D_U, \Omega')$ and $\rho(H_U, \Omega')$ are the nominal values of the lateral density stipulated at the depth $D_U(\Omega')$ and at the height $H_U(\Omega')$ of the upper bound of volumetric layer, respectively. The term $\beta(\Omega') \sum_{i=1}^{I} a_i(\Omega') \left(\mathrm{R} - r'\right)^i$ describes the radial density variation within the volumetric mass layer at a location Ω'. Alternatively, when modelling the gravitational field of the anomalous density structures of the Earth's interior, the density contrast $\Delta\rho(r', \Omega')$ of the volumetric mass layer relative to the (constant) reference background density ρ is defined as follows

$$\Delta\rho\left(r', \Omega'\right) = \rho - \rho\left(r', \Omega'\right) = \Delta\rho\left(D_U, \Omega'\right) - \beta\left(\Omega'\right) \sum_{i=1}^{I} a_i\left(\Omega'\right) \left(\mathrm{R} - r'\right)^i \quad \left[\mathrm{R} - D_U\left(\Omega'\right) \geq r' \geq \mathrm{R} - D_L\left(\Omega'\right) : \Omega' \in \Phi\right], \tag{5}$$

where $\Delta\rho(D_U, \Omega')$ is the nominal value of the lateral density contrast. *Tenzer et al. (2009)* adopted, for instance, the reference crust density of $\rho^{\mathrm{crust}} = 2670\,\mathrm{kg\,m^{-3}}$ in computing the stripping gravity corrections due to the anomalous density structures within the Earth's crust.

3. Gravitational field of laterally distributed and radially varying mass density layer

For the mass density model in Eq. (3), the expression for computing the gravitational potential V generated by the volumetric layer with a variable depth and thickness having a laterally distributed radial density variation is found to be (see Appendix I, Eq. I.10)

$$V(r,\Omega) = \frac{\mathrm{GM}}{\mathrm{R}} \sum_{n=0}^{\bar{n}} \sum_{m=-n}^{n} \left(\frac{\mathrm{R}}{r}\right)^{n+1} V_{\mathrm{n,m}}\, Y_{\mathrm{n,m}}(\Omega). \tag{6}$$

The corresponding gravitational attraction g (defined approximately as a negative radial derivative of the respective potential V reads

$$g(r,\Omega) \cong -\frac{\partial V(r,\Omega)}{\partial r} = \\ = \frac{\mathrm{GM}}{\mathrm{R}^2} \sum_{n=0}^{\bar{n}} \sum_{m=-n}^{n} \left(\frac{\mathrm{R}}{r}\right)^{n+2} (n+1)\, V_{\mathrm{n,m}}\, Y_{\mathrm{n,m}}(\Omega), \tag{7}$$

where $\mathrm{GM} = 3986005 \times 10^{8}\,\mathrm{m^3 s^{-2}}$ is the geocentric gravitational constant (defined as $\mathrm{GM} = (4\pi/3)\,\mathrm{GR}^3\, \bar{\rho}^{\mathrm{Earth}}$, where $\bar{\rho}^{\mathrm{Earth}} = 5500\,\mathrm{kg\,m^{-3}}$ is the adopted value of the Earth's mean mass density), $Y_{\mathrm{n,m}}$ are the (fully normalised) surface spherical harmonic functions, and $\bar{n}$ is the upper summation index of spherical harmonics. The coefficients $V_{\mathrm{n,m}}$ in Eqs. (6) and (7) read

$$V_{\mathrm{n,m}} = \frac{3}{2n+1} \frac{1}{\bar{\rho}^{\mathrm{Earth}}} \sum_{i=0}^{I} \left(Fl_{\mathrm{n,m}}^{(\mathrm{i})} - Fu_{\mathrm{n,m}}^{(\mathrm{i})} \right). \tag{8}$$

The numerical coefficients $\{Fl_{\mathrm{n,m}}^{(\mathrm{i})}, Fu_{\mathrm{n,m}}^{(\mathrm{i})} : \ i = 0, 1, ..., I\}$ are defined as follows

$$Fl_{\mathrm{n,m}}^{(\mathrm{i})} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1+i} \frac{L_{\mathrm{n,m}}^{(k+1+i)}}{\mathrm{R}^{\mathrm{k+1}}}, \tag{9}$$

and

$$Fu_{\mathrm{n,m}}^{(\mathrm{i})} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1+i} \frac{U_{\mathrm{n,m}}^{(k+1+i)}}{\mathrm{R}^{\mathrm{k+1}}}. \tag{10}$$

The convergence and optimal truncation of binomial series in Eqs. (9) and (10) were studied in detail by *Rummel et al. (1988)* and *Sun and Sjöberg (2001)*. The terms $\sum_{m=-n}^{n} L_{n,m} Y_{n,m}$ and $\sum_{m=-n}^{n} U_{n,m} Y_{n,m}$ define the spherical lower-bound and upper-bound laterally distributed radial density variation functions L_n and U_n of degree n. These spherical functions and their higher-order terms $\{L_n^{(k+1+i}, U_n^{(k+1+i)} : \ k = 0, 1, ... ; \ i = 1, 2, ..., I\}$ are defined as follows

$$L_n^{(k+1+i)}(\Omega) = \begin{cases} \frac{4\pi}{2n+1} \iint\limits_{\Phi} \rho(D_U, \Omega') \ D_L^{k+1}(\Omega') P_n(\cos\psi) \ d\Omega' \\ = \sum\limits_{m=-n}^{n} L_{n,m}^{(k+1)} Y_{n,m}(\Omega) & i = 0 \\ \\ \frac{4\pi}{2n+1} \iint\limits_{\Phi} \beta(\Omega') \ a_i(\Omega') \ D_L^{k+1+i}(\Omega') P_n(\cos\psi) \ d\Omega' \\ = \sum\limits_{m=-n}^{n} L_{n,m}^{(k+1+i)} Y_{n,m}(\Omega) & i = 1, 2, ..., I \end{cases} \tag{11}$$

and

$$U_n^{(k+1+i)}(\Omega) = \begin{cases} \frac{4\pi}{2n+1} \iint\limits_{\Phi} \rho(D_U, \Omega') \ D_U^{k+1}(\Omega') P_n(\cos\psi) \ d\Omega' \\ = \sum\limits_{m=-n}^{n} U_{n,m}^{(k+1)} Y_{n,m}(\Omega) & i = 0 \\ \\ \frac{4\pi}{2n+1} \iint\limits_{\Phi} \beta(\Omega') \ a_i(\Omega') \ D_U^{k+1+i}(\Omega') P_n(\cos\psi) \ d\Omega' \\ = \sum\limits_{m=-n}^{n} U_{n,m}^{(k+1+i)} Y_{n,m}(\Omega) & i = 1, 2, ..., I \end{cases} \tag{12}$$

where P_n are the Legendre polynomials of degree n for the argument of cosine of the spherical distance ψ. The coefficients $L_{n,m}$ and $U_{n,m}$ combine information on the geometry and density distribution of volumetric layer. The coefficients $L_{n,m}$ and $U_{n,m}$ are generated to a certain degree of spherical harmonics using the discrete data of the spatial density distribution (i.e., typically provided by means of density, depth and thickness data) of a particular structural component of the Earth's interior. Since depths and heights in Eqs. (1) and (2) are defined positive with respect to the geoid surface, the coefficients $V_{\mathrm{n,m}}$ in Eq. (8) are multiplied by -1 when the volumetric layer is situated above the geoid surface. Consequently, the spherical functions $\mathrm{L_n}$ and $\mathrm{U_n}$ of degree n in Eqs. (11) and (12) are defined for the heights H_L and H_U of the lower and upper bounds of the volumetric mass layer situated above the geoid surface. Since the summation in Eqs. (6) and (7) is finite the validation of the expressions for computing the gravitational field quantities is not restricted to the outer space of the Brillouin sphere (cf. *Vaníček et al., 1995*). We note that the expressions in Eqs. (6–12) can directly be used if the volumetric mass layer is distributed above and below the geoid surface using only one set of the coefficients $\mathrm{L_{n,m}}$ and $\mathrm{U_{n,m}}$ for describing the geometry of the lower and upper bound of this volumetric layer.

The generic expressions in Eqs. (6–12) are further specified for the radial, lateral and homogeneous density distribution models commonly adopted for a representation of particular subsurface density structures.

4. Gravitational field of radially varying mass density layer

When assuming the radially varying density model (without lateral changes) within the volumetric layer, the density function in Eq. (3) takes the following form

$$\rho\left(r'\right) = \rho\left(D_U\right) + \beta \sum_{i=1}^{I} a_i \left(\mathrm{R} - r'\right)^i$$
$$\left[\,\mathrm{R} - D_U\left(\Omega'\right) \geq r' \geq \mathrm{R} - D_L\left(\Omega'\right) : \Omega' \in \Phi\,\right], \tag{13}$$

where $\rho(D_U)$ is the nominal (constant) value of the mass density at the depth $D_U(\Omega')$ of the upper bound of volumetric layer, and the constant values of the parameters $\rho(D_U)$, β, and $\{a_i : i = 1, 2, ..., I\}$ describe the radial density variation within the whole volumetric layer. The spherical lower-bound and upper-bound radially varying mass density functions $\mathrm{L_n}$ and $\mathrm{U_n}$ of degree n and their higher-order terms $\{L_{\mathrm{n}}^{(\mathrm{k+1+i})}, U_{\mathrm{n}}^{(\mathrm{k+1+i})} :\ k = 0, 1, ... ;\ i = 1, 2,\ ..., I\}$ are defined as

$$L_{\mathrm{n}}^{(k+1+i)}(\Omega) = \begin{cases} \rho(D_U)\,\frac{4\pi}{2n+1} \iint\limits_{\Phi} D_L^{k+1}(\Omega')\,P_n(\cos\psi)\,\mathrm{d}\Omega' \\ = \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i=0 \\ \\ \beta\, a_i \frac{4\pi}{2n+1} \iint\limits_{\Phi} D_L^{k+1+i}(\Omega')\,P_n(\cos\psi)\,\mathrm{d}\Omega' \\ = \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1+i)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i=1,\,2,\,...,\,I \end{cases} \tag{14}$$

and

$$U_{\mathrm{n}}^{(k+1+i)}(\Omega) = \begin{cases} \rho(D_U)\,\frac{4\pi}{2n+1} \iint\limits_{\Phi} D_U^{k+1}(\Omega')\,P_n(\cos\psi)\,\mathrm{d}\Omega' \\ = \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i=0 \\ \\ \beta\, a_i \frac{4\pi}{2n+1} \iint\limits_{\Phi} D_U^{k+1+i}(\Omega')\,P_n(\cos\psi)\,\mathrm{d}\Omega' \\ = \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1+i)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i=1,\,2,\,...,\,I \end{cases} \tag{15}$$

5. Gravitational field of laterally varying mass density layer

When assuming only the lateral density variations within the volumetric layer, the numerical coefficients $V_{\mathrm{n,m}}$ in Eq. (8) become

$$V_{\mathrm{n,m}} = \frac{3}{2n+1}\frac{1}{\bar{\rho}^{\mathrm{Earth}}}\left(Fl_{\mathrm{n,m}} - Fu_{\mathrm{n,m}}\right), \tag{16}$$

where $Fl_{\mathrm{n,m}}$ and $Fu_{\mathrm{n,m}}$ are given by

$$Fl_{\mathrm{n,m}} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1} \frac{L_{\mathrm{n,m}}^{(k+1)}}{\mathrm{R}^{\mathrm{k+1}}}, \tag{17}$$

and

$$Fu_{\mathrm{n,m}} = \sum_{k=0}^{n+2} \binom{n+2}{k} \frac{(-1)^k}{k+1} \frac{U_{\mathrm{n,m}}^{(k+1)}}{\mathrm{R}^{\mathrm{k+1}}}. \tag{18}$$

The spherical lower-bound and upper-bound lateral density variation functions $\mathrm{L_n}$ and $\mathrm{U_n}$ of degree n and their higher-order terms $\{L_{\mathrm{n}}^{(\mathrm{k+1})}, U_{\mathrm{n}}^{(\mathrm{k+1})} : k = 1, 2, ...\}$ are given by

$$\begin{aligned} L_{\mathrm{n}}^{(k+1)}(\Omega) &= \frac{4\pi}{2n+1} \iint\limits_{\Phi} \rho(\Omega')\; D_L^{k+1}(\Omega')\, P_n(\cos\psi)\; \mathrm{d}\Omega' = \\ &= \sum_{\mathrm{m=-n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega)\,, \end{aligned} \tag{19}$$

and

$$\begin{aligned} U_{\mathrm{n}}^{(k+1)}(\Omega) &= \frac{4\pi}{2n+1} \iint\limits_{\Phi} \rho(\Omega')\; D_U^{k+1}(\Omega')\, P_n(\cos\psi)\; \mathrm{d}\Omega' = \\ &= \sum_{\mathrm{m=-n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega)\,. \end{aligned} \tag{20}$$

6. Gravitational field of homogeneous mass density layer

Assuming the constant density ρ within the volumetric layer, the numerical coefficients $V_{\mathrm{n,m}}$ in Eq. (8) read

$$V_{\mathrm{n,m}} = \frac{3}{2n+1} \frac{\rho}{\bar{\rho}^{\mathrm{Earth}}} \left(Fl_{\mathrm{n,m}} - Fu_{\mathrm{n,m}} \right). \tag{21}$$

Consequently, the functions $\mathrm{L_n}$ and $\mathrm{U_n}$ of degree n and their higher-order terms $\{L_{\mathrm{n}}^{(\mathrm{k+1})}, U_{\mathrm{n}}^{(\mathrm{k+1})} : k = 0, 1, ...\}$ are defined as

$$\begin{aligned} L_{\mathrm{n}}^{(k+1+i)}(\Omega) &= \frac{4\pi}{2n+1} \iint\limits_{\Phi} D_L^{k+1}(\Omega')\, P_n(\cos\psi)\; \mathrm{d}\Omega' = \\ &= \sum_{\mathrm{m=-n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega)\,, \end{aligned} \tag{22}$$

and

$$U_{\mathrm{n}}^{(k+1+i)}(\Omega) = \frac{4\pi}{2n+1} \iint\limits_{\Phi} D_U^{k+1}(\Omega')\, P_n(\cos\psi)\, \mathrm{d}\Omega' = \\ = \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega)\,. \tag{23}$$

The terms $\sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \mathrm{L}_{\mathrm{n,m}}\, \mathrm{Y}_{\mathrm{n,m}}$ and $\sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \mathrm{U}_{\mathrm{n,m}}\, \mathrm{Y}_{\mathrm{n,m}}$ in this case define the spherical lower-bound and upper-bound functions L_{n} and U_{n} of degree n, and the associated coefficients $\mathrm{L}_{\mathrm{n,m}}$ and $\mathrm{U}_{\mathrm{n,m}}$ describe the geometry of the homogeneous density mass layer.

7. Examples of modelling the crust density structures

The expressions from Section 4, which define the gravitational field generated by the volumetric layer with a radially varying mass density distribution model, can be utilised in computing the bathymetric (ocean density contrast) stripping gravity corrections. Since the geoid surface represents the oceanic upper bound, the numerical coefficients $V_{\mathrm{n,m}}$ in Eq. (8) are computed using the following expression

$$V_{\mathrm{n,m}} = \frac{3}{2n+1} \frac{1}{\bar{\rho}^{\mathrm{Earth}}} \sum_{i=0}^{I} \sum_{m=-n}^{n} Fl_{\mathrm{n,m}}^{(\mathrm{i})}. \tag{24}$$

The coefficients $\{Fl_{\mathrm{n,m}}^{(\mathrm{i})} : \; i = 0, 1, ..., I\}$ are defined in Eq. (9), and the function L_{n} of degree n and their higher-order terms $\{L_{\mathrm{n}}^{(\mathrm{k+1+i})} : \; k = 0, 1, ... ; \; i = 1, 2, ..., I\}$ are given by

$$L_{\mathrm{n}}^{(k+1+i)}(\Omega) = \begin{cases} \Delta\rho \frac{4\pi}{2n+1} \iint\limits_{\Phi} D_L^{k+1}(\Omega')\, P_n(\cos\psi)\, \mathrm{d}\Omega' \\ = \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i = 0 \\ \\ \beta\, a_i \frac{4\pi}{2n+1} \iint\limits_{\Phi} D_L^{k+1+i}(\Omega')\, P_n(\cos\psi)\, \mathrm{d}\Omega' \\ = \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1+i)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) & i = 1, 2, ..., I \end{cases} \tag{25}$$

where $\Delta\rho$ is a nominal value of the density contrast. The coefficients $\mathrm{L}_{\mathrm{n,m}}$ in Eq. (25) describe the geometry of the ocean bottom relief. The approximation of the actual seawater density distribution by its mean value yields relative errors up to about 2% in computed quantities of the gravitational field. To reduce these errors, *Tenzer et al. (2011a)* facilitated a depth-dependent seawater density model in deriving expressions for computing the bathymetric stripping gravity corrections. They demonstrated that the approximation of the seawater density by the depth-dependent density model reduces the maximum errors to less than 0.6%. The corresponding depth-averaged errors are below 0.1%. They defined the nominal value of the ocean density contrast $\Delta\rho_0^{\mathrm{w}}$ as the difference between the reference values of the crustal density ρ^{crust} and the seawater density ρ_0^{w}; i.e., $\Delta\rho_0^{\mathrm{w}} = \rho^{\mathrm{crust}} - \rho_0^{\mathrm{w}}$. The value of surface seawater density $\rho_0^{\mathrm{w}} = 1027.91\mathrm{kg/m^3}$ (cf. *Gladkikh and Tenzer, 2011*) was adopted as the reference seawater density. For the adopted value of the reference crustal density ρ^{crust} of $2670\,\mathrm{kg/m^3}$ (cf. *Hinze, 2003*), the reference ocean density contrast (at zero depth) equals $\Delta\rho_0^{\mathrm{w}} = 1642.09\,\mathrm{kg/m^3}$. The parameters of the depth-dependent density term in Eq. (25) up to the second degree ($I = 2$) are given by the following values (*Tenzer et al., 2011a*): $\beta = 0.00637\,\mathrm{kg/m^3}$, $a_1 = 0.7595\,\mathrm{m^{-1}}$, and $a_2 = -4.3984 \times 10^{-6}\,\mathrm{m^{-2}}$. These values were estimated from the oceanographic data of the World Ocean Atlas 2009 (provided by NOAA's National Oceanographic Data Center; *Johnson et al., 2009*) and the World Ocean Circulation Experiment 2004 (provided by the German Federal Maritime and Hydrographic Agency; *Gouretski and Koltermann, 2004*).

The global bathymetric model coefficients $\mathrm{L}_{\mathrm{n,m}}$ in Eq. (25) with a spectral resolution complete to degree 2160 of spherical harmonics can be generated using the coefficients of the global topographic/bathymetric model DTM2006.0 (*Pavlis et al., 2007*) and the global geopotential model EGM 2008 (*Pavlis et al., 2008*). The EGM2008 geoid coefficients are calculated according to the Bruns formula (see e.g., *Heiskanen and Moritz, 1967*). The DTM2006.0 coefficients describe the global geometry of the topographic heights above mean sea level (MSL) which are reckoned positive, and the bathymetric depths below MSL which are reckoned negative. The global topographic/bathymetric model DTM2006.0 was released together with EGM2008 by the U.S. National Geospatial-Intelligence Agency EGM development team.

Tenzer et al. (2011b) used the expressions from Section 5 formulated for the laterally varying mass density contrast layer with varying depth and thickness in the gravimetric forward modelling of the anomalous density structures within the Earth's solid crust (i.e., excluding the ocean density contrast). They used the 2 × 2 arc-deg global discrete data of density, depth, and thickness of the (soft and hard) sediments and consolidated (upper, middle, and lower) crust components from the global crustal model CRUST2.0 (*Bassin et al., 2000*) to generate the coefficients $\mathrm{L}_{\mathrm{n,m}}$ and $\mathrm{U}_{\mathrm{n,m}}$ according to Eqs. (19) and (20). These coefficients were then used for computing the corresponding gravitational field quantities with a low spectral resolution complete to degree 90 of spherical harmonics. The density contrasts were defined relative to the reference crustal density of 2670 kg m^{-3}. The expressions derived in Section 3 can be utilised to model globally the sediments density contrast with the laterally distributed depth-dependent density variations once the global crustal models which incorporate these density data become available.

When the homogeneous mass density contrast layer with a variable height and thickness is situated above the geoid surface the numerical coefficients $V_{\mathrm{n,m}}$ in Eq. (21) are computed as follows

$$V_{\mathrm{n,m}} = \frac{3}{2n+1} \frac{\Delta\rho}{\bar{\rho}^{\mathrm{Earth}}} \left(Fu_{\mathrm{n,m}} - Fl_{\mathrm{n,m}} \right), \tag{26}$$

where $\Delta\rho$ is the (constant) density contrast of the homogeneous volumetric mass layer. The spherical lower-bound and upper-bound functions L_{n} and U_{n} of degree n and their higher-order terms $\{L_{\mathrm{n}}^{(\mathrm{k+1})}, U_{\mathrm{n}}^{(\mathrm{k+1})} : k = 1, 2, ...\}$ are defined for the heights H_L and H_U in the following form

$$\left. \begin{matrix} U_{\mathrm{n}}^{(k+1)}(\Omega) \\ L_{\mathrm{n}}^{(k+1)}(\Omega) \end{matrix} \right\} = \frac{4\pi}{2n+1} \iint_{\Phi} P_n(\cos\psi) \left\{ \begin{matrix} H_L^{k+1}(\Omega') \\ H_U^{k+1}(\Omega') \end{matrix} \right\} \mathrm{d}\Omega' =$$

$$= \begin{cases} \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} U_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) \\ \sum\limits_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} L_{\mathrm{n,m}}^{(k+1)} \mathrm{Y}_{\mathrm{n,m}}(\Omega) \end{cases} . \tag{27}$$

Tenzer et al. (2010a) used the 10 × 10 arc-min mean topographic heights computed by spatial averaging of the 30 × 30 arc-sec global elevation data from GTOPO30 (provided by the US Geological Survey's EROS Data Center) and the 10 × 10 arc-min continental ice-thickness data from ICE-5G (VM2 L90) made available by *Peltier (2004)* to generate the coefficients $L_{\mathrm{n,m}}$ and $U_{\mathrm{n,m}}$ complete to degree 180 of spherical harmonics according to Eq. (27). These coefficients were then used to compute the ice density contrast stripping gravity corrections. The ice density contrast $\Delta\rho^{\mathrm{ice}}$ was defined as the difference between the reference density values of the crust ρ^{crust} and glacial ice ρ^{ice} i.e., $\Delta\rho^{\mathrm{ice}} = \rho^{\mathrm{crust}} - \rho^{\mathrm{ice}}$. For the adopted values of the reference crustal density $2670\,\mathrm{kg\,m^{-3}}$ and the density of glacial ice $917\,\mathrm{kg\ m^{-3}}$ (cf. *Cutnell and Kenneth, 1995*) the ice density contrast equals $1753\,\mathrm{kg\,m^{-3}}$.

When the geoid surface represents the lower bound of the homogeneous mass density layer, the numerical coefficients $V_{\mathrm{n,m}}$ in Eq. (26) become

$$V_{\mathrm{n,m}} = \frac{3}{2n+1}\frac{\rho}{\bar{\rho}^{\mathrm{Earth}}} F u_{\mathrm{n,m}}, \tag{28}$$

where the coefficients $U_{\mathrm{n,m}}$ are computed according to Eq. (27). These expressions were used, for instance, by *Novák (2010)* to compute the topographic gravity correction adopting the constant value of the reference crustal density of $2670\,\mathrm{kg\,m^{-3}}$. The coefficients $U_{\mathrm{n,m}}$ in Eqs. (27) which describe the geometry of the global topography can be generated from the coefficients of the global topographic/bathymetric model DTM2006.0 and the coefficients of the EGM2008 global geoid model complete to degree 2160 of spherical harmonics.

8. Summary and concluding remarks

We have derived the generic expressions for the gravitational potential and attraction generated by an arbitrary volumetric layer with a variable depth and thickness having a laterally distributed radial density variation. These expressions utilise the functions $\mathrm{L_n}$ and $\mathrm{U_n}$ which combine information on the geometry and density distribution of a volumetric mass layer. The generic expressions were further specified for the radial, lateral, and homogeneous density distribution models commonly adopted for a representation of particular subsurface density structures.

We have demonstrated that the generic expressions can uniformly be applied in the gravimetric forward modelling of all major known density structures within the Earth's crust. The homogeneous density mass layer was used to represent the ice density contrast and the topography of homogeneous density (with the geoid surface representing the lower-bound of volumetric mass layer). The radially varying density mass layer was used to model the ocean density contrast (with the geoid surface representing the upper-bound of volumetric mass layer). The currently available global elevation and bathymetry data (from the global topographic/bathymetric model DTM2006.0) allow modelling the topography and ocean density contrast to a very high spectral resolution up to the spherical harmonic degree 2160. A spatial resolution of modelling the ice density contrast of 10×10 arc-min is possible based on the currently available global data of continental ice thickness. The best currently available global crustal model CRUST2.0 provides information on density distribution within sediments and remaining crust with a 2×2 arc-deg spatial resolution. The sediments and consolidated crust components density contrast structures were represented by the laterally varying density mass layers. The CRUST2.0 soft sediments vary in density from 1700 to 2300 kg/m^3 and reach a maximum thickness of about 2 km, while the CRUST2.0 hard sediments vary between 2300 and 2600 kg/m^3 and become up to 18 km thick at places. The approximation of the soft and hard sediment components and their density variability by two individual laterally varying density mass layers thus reflect to a certain degree the increasing density of sediments with depth due to compaction. However, the approximation of the sediment spatial density distribution can further be improved adopting the laterally distributed and depth-dependent density variation model once more accurate global sediment data with a higher resolution become available. Depending on data availability (typically seismic reflection data), a more accurate representation of the mantle lithosphere and sub-lithospheric mantle can be achieved utilising the depth and lateral density distribution models. Current models typically use only simple models based on the assumption of a spherically symmetric density distribution.

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References

Artemjev M. E., Kaban M. K., Kucherinenko V. A., Demjanov G. V., Taranov V. A., 1994: Subcrustal density inhomogeneities of the Northern Euroasia as derived from the gravity data and isostatic models of the lithosphere. Tectonophysics, **240**, 248–280.

Bassin C., Laske G., Masters G., 2000: The current limits of resolution for surface wave tomography in North America. EOS, Trans AGU, **81**, F897.

Cutnell J. D., Kenneth W. J., 1995: Physics. 3rd Edition. Wiley, New York.

Gladkikh V., Tenzer R., 2011: A mathematical model of the global ocean saltwater density distribution. Pure Appl. Geophys., doi: 10.1007/s00024-011-0275-5.

Gouretski V. V., Koltermann K. P., 2004: Berichte des Bundesamtes für Seeschifffahrt und Hydrographie. No. 35.

Heiskanen W. H., Moritz H., 1967: Physical geodesy. San Francisco, W. H., Freeman and Co.

Hinze W. J., 2003: Bouguer reduction density, why 2.67? Geophysics, **68**, 5, 1559–1560.

Hobson E. W., 1931: The Theory of Spherical and Ellipsoidal Harmonics. Cambridge Univ. Press, republished 1955 by Chelsea Publishing Comp., New York.

Johnson D. R., Garcia H. E., Boyer T. P., 2009: World Ocean Database 2009 Tutorial. (ed.) Levitus S., NODC Internal Report 21, NOAA Printing Office, Silver Spring, MD, 18 p.

Moritz H., 1980: Advanced Physical Geodesy. Abacus Press, Tunbridge Wells.

Novák P., 2010: High resolution constituents of the Earth gravitational field. Surv. Geoph., **31**, 1, 1–21.

Pavlis N. K., Rapp R. H., 1990: The development of an isostatic gravitational model to degree 360 and its use in global gravity modelling. Geophys. J. Int., **100**, 369–378.

Pavlis N. K., Factor J. K., Holmes S. A., 2007: Terrain-Related Gravimetric Quantities Computed for the Next EGM. In: Gravity Field of the Earth. A. Kiliçoglu and R. Forsberg (Eds.), Proceedings of the 1st International Symposium of the International Gravity Field Service (IGFS), Harita Dergisi, Special Issue, **18**, General Command of Mapping, Ankara, Turkey.

Pavlis N. K., Holmes S. A., Kenyon S. C., Factor J. K., 2008: An Earth Gravitational Model to Degree 2160: EGM 2008. Presented at Session G3: "GRACE Science Applications", EGU Vienna.

Peltier W. R., 2004: Global Glacial Isostasy and the Surface of the Ice-Age Earth: The ICE-5G (VM2) Model and GRACE. Ann. Rev. Earth and Planet. Sci., **32**, 111–149.

Rummel R., Rapp R. H., Sünkel H., Tscherning C. C., 1988: Comparison of global topographic/isostatic models to the Earth's observed gravitational field. Report, **388**, The Ohio State University, Columbus, Ohio 43210–1247.

Sjöberg L. E., 1998: The exterior Airy/Heiskanen topographic-isostatic gravity potential anomaly and the effect of analytical continuation in Stokes' formula. J. Geod., **72**, 654–662.

Sjöberg L. E., Nahavandchi H., 2000: The atmospheric geoid effects in Stokes formula. Geoph. J. Int., **140**, 95–100.

Sun W., Sjöberg L. E., 2001: Convergence and optimal truncation of binomial expansions used in isostatic compensations and terrain corrections. J. Geod., **74**, 627–636.

Sünkel H., 1968: Global topographic-isostatic models. In: Mathematical and numerical techniques in Physical geodesy. (ed.) Sünkel H., Lecture Notes in Earth Sciences, **7**, Springer-Verlag, 417–462.

Tenzer R., Hamayun, Vajda P., 2009: Global maps of the CRUST 2.0 crustal components stripped gravity disturbances. J. Geophys. Res., **114**, B, 05408.

Tenzer R., Abdalla A., Vajda P., Hamayun, 2010a: The spherical harmonic representation of the gravitational field quantities generated by the ice density contrast. Contrib. Geophys. Geod., **40**, 3, 207–223.

Tenzer R., Vajda P., Hamayun, 2010b: A mathematical model of the bathymetry-generated external gravitational field. Contrib. Geophys. Geod., **40**, 1, 31–44.

Tenzer R., Novák P., Gladkikh V., 2011a: On accuracy of the bathymetry-generated gravitational field quantities for a depth-dependent seawater density distribution. Stud. Geophys. Geodaet., doi: 10.1007/s11200-010-0074-y.

Tenzer R., Novák P., Hamayun, Vajda P., 2011b: Spectral expressions for modelling the gravitational field of the Earth's crust density structure. Stud. Geophys. Geodaet., doi: 10.1007/s11200-011-0023-4.

Vaníček P., Najafi M., Martinec Z., Harrie L., Sjöberg L. E., 1995: Higher-degree reference field in the generalised Stokes-Helmert scheme for geoid computation. J. Geod., **70**, 176–182.

Appendix I: Spectral representation of potential

To derive the expression for the gravitational potential V in the spectral representation, Eq. (1) is first rewritten as

$$V\left(r,\Omega\right) = \mathrm{G} \iint\limits_{\Phi} \int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}} \rho\left(r',\Omega'\right)\ \ell^{-1}\left(r,\psi,r'\right)\ r'^2 \mathrm{d}r'\,\mathrm{d}\Omega' - \\ - \mathrm{G} \iint\limits_{\Phi} \int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}} \rho\left(r',\Omega'\right)\ \ell^{-1}\left(r,\psi,r'\right)\ r'^2 \mathrm{d}r'\,\mathrm{d}\Omega'. \qquad \text{(I.1)}$$

The first constituent on the right-hand side of Eq. (I.1) is the gravitational contribution generated by the volumetric density mass enclosed between the lower bound $\{D_L(\Omega') : \Omega' \in \Phi\}$ and the reference sphere of radius

R. The second constituent represents the gravitational contribution generated by the volumetric density mass enclosed between the upper bound $\{D_U(\Omega') : \Omega' \in \Phi\}$ and the reference sphere of radius R. Inserting the laterally distributed radial density variation model $\rho(D_U, \Omega')$ Eq. (3) to Eq. (I.1), we get

$$\begin{aligned}
V(r,\Omega) = {} & \mathrm{G} \iint\limits_{\Phi} \rho\left(D_U,\Omega'\right) \int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}} \ell^{-1}\left(r,\psi,r'\right)\, r'^2 \mathrm{d}r'\, \mathrm{d}\Omega' - \\
& - \mathrm{G} \iint\limits_{\Phi} \rho\left(D_U,\Omega'\right) \int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}} \ell^{-1}\left(r,\psi,r'\right)\, r'^2 \mathrm{d}r'\, \mathrm{d}\Omega' + \\
& + \mathrm{G} \iint\limits_{\Phi} \beta\left(\Omega'\right) \sum_{i=1}^{I} a_i\left(\Omega'\right) \times \\
& \times \int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}} \ell^{-1}\left(r,\psi,r'\right) \left(\mathrm{R}-r'\right)^i\, r'^2 \mathrm{d}r'\, \mathrm{d}\Omega' - \\
& - \mathrm{G} \iint\limits_{\Phi} \beta\left(\Omega'\right) \sum_{i=1}^{I} a_i\left(\Omega'\right) \times \\
& \times \int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}} \ell^{-1}\left(r,\psi,r'\right) \left(\mathrm{R}-r'\right)^i\, r'^2 \mathrm{d}r'\, \mathrm{d}\Omega'.
\end{aligned} \tag{I.2}$$

The spectral representation of the reciprocal spatial distance ℓ^{-1} for the external convergence domain $r \geq r'$ ($r \geq \mathrm{R} \wedge r' \leq \mathrm{R}$) is given by (e.g., *Hobson, 1931*)

$$\ell^{-1}\left(r,\psi,\, r'\right) = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n\left(\cos\psi\right). \tag{I.3}$$

The series in Eq. (I.3) is uniformly convergent for $r \geq r'$. Substituting the fundamental harmonic function in Eq. (I.3) to Eq. (I.2), we arrive at

$$\begin{aligned}
V(r,\Omega) = {} & \mathrm{G} \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n+1} \iint\limits_{\Phi} \rho\left(D_U,\Omega'\right) P_n\left(\cos\psi\right) \int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}} r'^{n+2} \mathrm{d}r'\ \mathrm{d}\Omega' - \\
& - \mathrm{G} \sum_{n=0}^{\infty} \left(\frac{1}{r}\right)^{n+1} \iint\limits_{\Phi} \rho\left(D_U,\Omega'\right) P_n\left(\cos\psi\right) \int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}} r'^{n+2} \mathrm{d}r'\ \mathrm{d}\Omega' +
\end{aligned}$$

$$+\,\mathrm{G}\sum_{n=0}^{\infty}\left(\frac{1}{r}\right)^{n+1}\iint\limits_{\Phi}\beta\left(\Omega'\right)\ \sum_{i=1}^{I}a_i\left(\Omega'\right)P_n\left(\cos\psi\right)\times$$
$$\times\int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}}\left(\mathrm{R}-r'\right)^i\ r'^{n+2}\mathrm{d}r'\ \ \mathrm{d}\Omega'-$$
$$-\,\mathrm{G}\sum_{n=0}^{\infty}\left(\frac{1}{r}\right)^{n+1}\iint\limits_{\Phi}\beta\left(\Omega'\right)\ \sum_{i=1}^{I}a_i\left(\Omega'\right)P_n\left(\cos\psi\right)\times$$
$$\times\int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}}\left(\mathrm{R}-r'\right)^i\ r'^{n+2}\mathrm{d}r'\ \ \mathrm{d}\Omega'. \tag{I.4}$$

Since the expansion of Newton's integral kernel converges uniformly when computed at locations outside the gravitating masses, the interchange of summation and integration in Eq. (I.4) is permissible (cf. *Moritz, 1980*). The application of the binomial theorem to the term r'^{n+2} in Eq. (I.4) yields

$$r'^{n+2}=\mathrm{R}^{n+2}\ \left(1-\frac{\mathrm{R}-r'}{\mathrm{R}}\right)^{n+2}\cong$$
$$\cong\mathrm{R}^{n+2}\sum_{k=0}^{n+2}\binom{n+2}{k}\left(\frac{\mathrm{R}-r'}{\mathrm{R}}\right)^k(-1)^k. \tag{I.5}$$

From Eq. (I.5), the solutions of the radial integrals in the first and second constituents on the right-hand side of Eq. (I.4) are found to be

$$\int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}}r'^{n+2}\mathrm{d}r'\cong\mathrm{R}^{n+2}\int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}}\sum_{k=0}^{n+2}\binom{n+2}{k}\left(\frac{\mathrm{R}-r'}{\mathrm{R}}\right)^k(-1)^k\,\mathrm{d}r'=$$
$$=\mathrm{R}^{n+3}\sum_{k=0}^{n+2}\binom{n+2}{k}\left[\frac{D_L\left(\Omega'\right)}{\mathrm{R}}\right]^{k+1}\frac{(-1)^k}{k+1}, \tag{I.6}$$

and

$$\int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}}r'^{n+2}\mathrm{d}r'\cong\mathrm{R}^{n+3}\sum_{k=0}^{n+2}\binom{n+2}{k}\left[\frac{D_U\left(\Omega'\right)}{\mathrm{R}}\right]^{k+1}\frac{(-1)^k}{k+1}. \tag{I.7}$$

Similarly, we have

$$\int_{\mathrm{R}-D_L(\Omega')}^{\mathrm{R}}\left(\mathrm{R}-r'\right)^i\ r'^{n+2}\mathrm{d}r'\ \cong$$

$$\cong \mathrm{R}^{n+3+i} \sum_{k=0}^{n+2} \binom{n+2}{k} \left[\frac{D_L\left(\Omega'\right)}{\mathrm{R}}\right]^{k+1+i} \frac{(-1)^k}{k+1+i}, \tag{I.8}$$

and

$$\int_{\mathrm{R}-D_U(\Omega')}^{\mathrm{R}} \left(\mathrm{R}-r'\right)^i \; r'^{n+2} \mathrm{d}r' \cong$$
$$\cong \mathrm{R}^{n+3+i} \sum_{k=0}^{n+2} \binom{n+2}{k} \left[\frac{D_U\left(\Omega'\right)}{\mathrm{R}}\right]^{k+1+i} \frac{(-1)^k}{k+1+i}. \tag{I.9}$$

The substitution from Eqs. (I.6-9) to Eq. (I.4) yields

$$\begin{aligned}
V\left(r,\Omega\right) = {} & \mathrm{G}\,\mathrm{R}^2 \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum_{k=0}^{n+2} \binom{n+2}{k} \left(\frac{1}{\mathrm{R}}\right)^{k+1} \frac{(-1)^k}{k+1} \times \\
& \times \iint_{\Phi} \rho\left(D_U,\Omega'\right) \, D_L^{k+1}\left(\Omega'\right) \, P_n\left(\cos\psi\right) \mathrm{d}\Omega' - \\
& - \mathrm{G}\,\mathrm{R}^2 \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum_{k=0}^{n+2} \binom{n+2}{k} \left(\frac{1}{\mathrm{R}}\right)^{k+1} \frac{(-1)^k}{k+1} \times \\
& \times \iint_{\Phi} \rho\left(D_U,\Omega'\right) \, D_U^{k+1}\left(\Omega'\right) \, P_n\left(\cos\psi\right) \mathrm{d}\Omega' + \\
& + \mathrm{G}\,\mathrm{R}^2 \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum_{i=1}^{I} \sum_{k=0}^{n+2} \binom{n+2}{k} \left(\frac{1}{\mathrm{R}}\right)^{k+1} \frac{(-1)^k}{k+1+i} \times \\
& \times \iint_{\Phi} \beta\left(\Omega'\right) \, a_i\left(\Omega'\right) \, D_L^{k+1+i}\left(\Omega'\right) \, P_n\left(\cos\psi\right) \mathrm{d}\Omega' - \\
& - \mathrm{G}\,\mathrm{R}^2 \sum_{n=0}^{\infty} \left(\frac{\mathrm{R}}{r}\right)^{n+1} \sum_{i=1}^{I} \sum_{k=0}^{n+2} \binom{n+2}{k} \left(\frac{1}{\mathrm{R}}\right)^{k+1} \frac{(-1)^k}{k+1+i} \times \\
& \times \iint_{\Phi} \beta\left(\Omega'\right) \, a_i\left(\Omega'\right) \, D_U^{k+1+i}\left(\Omega'\right) \, P_n\left(\cos\psi\right) \mathrm{d}\Omega'.
\end{aligned} \tag{I.10}$$