The forward problem of magnetometry for the oblate spheroid

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Abstract: We present analytical solution of the forward magnetometric problem for the oblate spheroid (rotational ellipsoid) as a causative body. The shorter semiaxis of the ellipsoid is supposed to be vertical to the surface of the earth. There is proved that the uniform inducing magnetic field $\boldsymbol{B}_0$ induces inside the spheroid also uniform magnetic field but its modulus and direction are different as compared to $\boldsymbol{B}_0$. The isolines and profile curves of ΔZ and ΔT are calculated on the plane $z =$ const above the ellipsoid, as well as on the surface of the hill in the shape of cutted cone.

Key words: geophysical magnetometry, potential due to oblate spheroid, magnetic anomalies due to magmatic bodies

1. Introduction

The solution of the forward magnetometric problem for the oblate rotational ellipsoid is interesting for the theoretical and also applied geophysical magnetometry. This causative body can be used also in some volcanic and post volcanic areas, e.g. as a model of laccolite. The magnetic problem for the triaxial ellipsoid is solved in numerous monographs, e.g. *Stratton (1941), Muratov (1976)* by using rather complicated elliptic integrals. We present here the solution by means of the method of separation of variables, which is more suitable for calculation of potential and namely of components of the anomalous field in comparison to the classical treatment. We solve our problem as similar problems of steady electric induction, e.g. in *Smythe (1968)* by using the method of separation of variables in the oblate spheroidal coordinates.

The axially symmetric oblate ellipsoid is the body which is bounded by surface of the second order degree, described by the equation

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$$(x^2 + y^2)/a^2 + z^2/b^2 = 1, \tag{1}$$

where $a(b)$ are major (minor) semiaxes of the ellipsoid, centered in the point $O \equiv (0, 0, 0)$. The cross-section of the ellipsoid by the plane $y = 0$ is depicted in Fig. 1, together with other parameters for our problem. We will calculate the magnetic induction anomaly considering magnetic permeability of the body to be uniform:

$$\mu_T = \mu_0(1 + \kappa), \tag{2}$$

where μ_0 is the magnetic permeability of vacuum $= 4\pi \times 10^{-7}$ Henry/m and κ is magnetic susceptibility of the ore filling the spheroid. It is known that for the hot magma in the magnetic chamber we have to put $\kappa \to 0$, but for cooled solidified one we put κ value for basalts or andesites $\kappa = 0.001 - 0.1$ (in SI system). Moreover, the solidified rock preserves thermoremanent magnetization which is as a rule 10 times greater than the magnetization $\boldsymbol{J}$ obtained due to induction. Let us note that similar magnetometry problem for prolate ellipsoid was solved in our previous paper *Hvoždara and Vozár (2010)*.

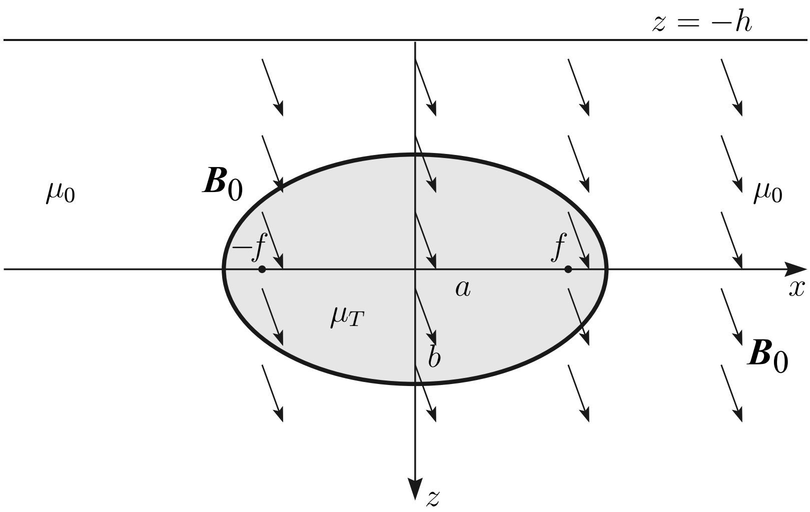


Fig. 1. The (x, z) section of the spheroid (gray) and parameters of the problem.

2. Formulation of the problem

Let us consider the prolate rotational spheroid embedded in the uniform magnetic field $\boldsymbol{B}_0$ with inclination angle I:

$$\boldsymbol{B}_0 \equiv (X_0, Y_0, Z_0), \tag{3}$$

where X_0, Y_0, Z_0 are components in the local coordinate systems. Due to rotational symmetry we can put the axis x in the direction of magnetic meridian, so $Y_0 = 0$ and the potential of the magnetic field $\boldsymbol{B}_0$ expressed as:

$$U_0 = -B_0(x \cos I + z \sin I). \tag{4}$$

Here B_0 is the modulus (total field) of the magnetic induction $\boldsymbol{B}_0$:

$$B_0 = \left(X_0^2 + Z_0^2\right)^{1/2}, \quad X_0 = B_0 c_I, \quad Z_0 = B_0 s_I, \tag{5}$$

where $c_I = \cos I$, $s_I = \sin I$. The magnetic field $\boldsymbol{B}$ in our model is steady in time so it obeys the Maxwell equations:

$$\operatorname{rot} \boldsymbol{B} = 0, \quad \operatorname{div} \boldsymbol{B} = 0, \tag{6}$$

hence it can be derived from the magnetic potential $U(x, y, z)$:

$$\boldsymbol{B} = -\operatorname{grad} U. \tag{7}$$

Note that we introduce the potential for the magnetic induction $\boldsymbol{B}$, while traditional treatment use the potential for the magnetic intensity $\boldsymbol{H} = \boldsymbol{B}/\mu$. It is clear that potential obeys the Laplace equation

$$\operatorname{div} \operatorname{grad} U = 0, \quad (\nabla^2 U = 0). \tag{8}$$

We denote the potential inside the spheroid by U_T and outside as $U_1 = U_0 + U_1^*$, where U_1^* is the perturbing potential outside the body.

The potential of the unperturbed magnetic field $\boldsymbol{B}_0 \equiv (X_0, 0, Z_0)$ far from the spheroid is:

$$U_0(x, y, z) = -B_0(c_I\, x + s_I\, z). \tag{9}$$

The presence of the spheroid causes outside spheroid the perturbation potential $U_1^*(x, y, z)$ which also obeys Laplace equation:

$$\nabla^2 U_1^*(x,y,z) = 0. \tag{10}$$

The magnetic potential in the interior of the spheroid is $U_T(x,y,z)$, which is also harmonic function. On the surface S of the spheroid we must have continuity of the tangential component of the intensity $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$ and normal component of the magnetic induction $\boldsymbol{B}$. For potentials this gives conditions:

$$\left[U_0 + U_1^*\right]_S = \mu_r^{-1}\left[U_T\right]_S, \quad \partial\left[U_0 + U_1^*\right]/\partial n|_S = \left[\partial U_T/\partial n\right]_S, \tag{11}$$

where $\mu_r = \mu_T/\mu_0 = 1 + \kappa$ is relative permeability of the spheroid. The methods of mathematical physics *(Morse and Feschbach, 1953; Arfken, 1966)* give very effective tools for solutions of the above potential problem by using the methods of separation of variables for the oblate spheroidal coordinate system (α, β, φ). These are linked to our Carthesian system (x,y,z):

$$x = f \operatorname{ch}\alpha \sin\beta \cos\varphi, \quad y = f \operatorname{ch}\alpha \sin\beta \sin\varphi, \quad z = f \operatorname{sh}\alpha \cos\beta, \tag{12}$$

(Madelung, 1957; Lebedev, 1963). The coordinates α, β, φ are from intervals $\alpha \in \langle 0, +\infty)$, $\beta \in \langle 0, \pi\rangle$, $\varphi \in \langle 0, 2\pi\rangle$ and f is the oblatness parameter

$$f = \sqrt{a^2 - b^2}, \tag{13}$$

i.e. f is the linear eccentricity of the generating ellipse. Note that we already used this method in *Hvoždara (2008)* for the mathematically similar groundwater flow problem.

From transformation equations (12) it can be derived that the coordinate surfaces $\alpha = \text{const}$ are oblate rotational ellipsoids

$$\frac{x^2 + y^2}{f^2 \operatorname{ch}^2\alpha} + \frac{z^2}{f^2 \operatorname{sh}^2\alpha} = 1, \quad \text{or} \quad \frac{r^2}{f^2 \operatorname{ch}^2\alpha} + \frac{z^2}{f^2 \operatorname{sh}^2\alpha} = 1, \tag{14}$$

where $r = \sqrt{x^2 + y^2}$ is distance from z axis. The equation of generating ellipse in the (x,z) plane for our spheroid is:

$$x^2/a^2 + z^2/b^2 = 1. \tag{15}$$

This is matched to the spheroid $\alpha = \alpha_0$ of the sets of spheroids (14) if we put:

$$a^2 = f^2 \operatorname{ch}^2\alpha_0, \quad b^2 = f^2 \operatorname{sh}^2\alpha_0. \tag{16}$$

We know that there holds property

$$\operatorname{ch}^2\alpha_0 - \operatorname{sh}^2\alpha_0 = 1, \tag{17}$$

so we easily find:

$$f^2 = a^2 - b^2, \quad f = \sqrt{a^2 - b^2}, \tag{18}$$

which confirms that f is linear excentricity of generating ellipse, it is the distance of foci from the ellipse centre as shown in Fig. 1. The polar axis for the angle β is $z \in \langle 0, +\infty)$; it corresponds to $\beta = 0$. The coordinate surfaces $\beta = \text{const}$ can be obtained from (12) by excluding $\operatorname{ch}\alpha$ and $\operatorname{sh}\alpha$ by using property $\operatorname{ch}^2\alpha - \operatorname{sh}^2\alpha = 1$. These are confocal rotational hyperboloids (see Fig. 2):

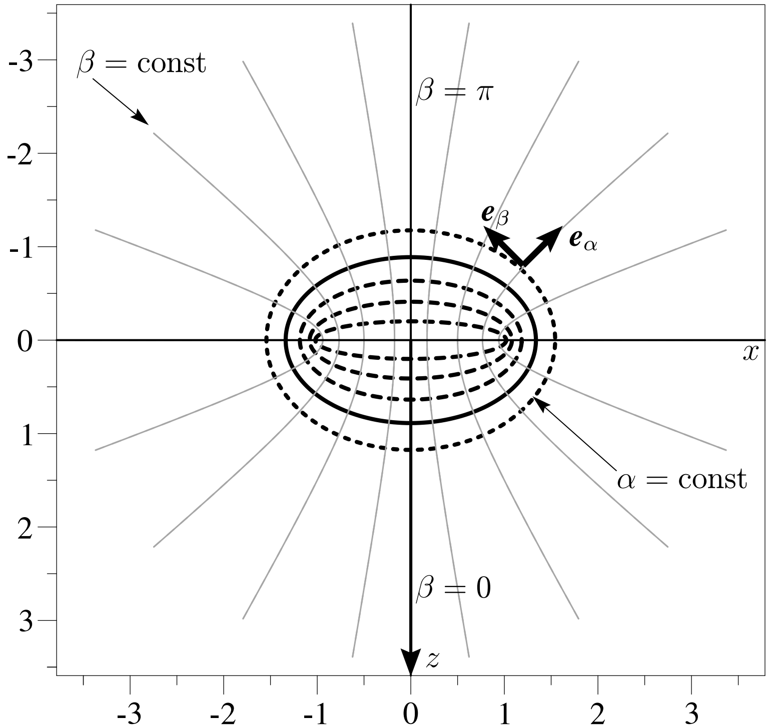


Fig. 2. The (x, z) section of coordinate surfaces $\alpha = \text{const}$ (ellipses), and $\beta = \text{const}$ (hyperboles).

$$\frac{r^2}{f^2\sin^2\beta} - \frac{z^2}{f^2\cos^2\beta} = 1. \tag{19}$$

It is necessary to note that the plane $z = 0$ corresponds to the surface $\alpha = 0$ and the circle $x^2 + y^2 = f^2$ is the focal circle. From relations (16) we also obtain:

$$e^{\alpha_0} = (a+b)/f, \quad \alpha_0 = \ln[(a+b)/f]. \tag{20}$$

In this manner we can link spheroidal coordinate system (α, β, φ) to the generating ellipse. We add that Lame's metrical parameters are as follows:

$$h_\alpha = f\sqrt{\mathrm{ch}^2\alpha - \sin^2\beta}, \quad h_\beta = h_\alpha, \quad h_\varphi = f\,\mathrm{ch}\,\alpha\sin\beta, \tag{21}$$

(see e.g. *Madelung, 1957*). The particular solution of Laplace equation in the system (α, β, φ) can be found in e.g. *Lebedev (1963)* in the form:

$$U_{mn}(\alpha,\beta,\varphi) = [M_{mn}\cos m\varphi + N_{mn}\sin m\varphi] \begin{Bmatrix} P_n^m(\mathrm{i}\,\mathrm{sh}\,\alpha) \\ Q_n^m(\mathrm{i}\,\mathrm{sh}\,\alpha) \end{Bmatrix} P_n^m(\cos\beta), \tag{22}$$

where $i = \sqrt{-1}$ is imaginary unit and $P_n^m(\mathrm{i}\,\mathrm{sh}\,\alpha)$, $Q_n^m(\mathrm{i}\,\mathrm{sh}\,\alpha)$ are associated Legendre functions of degree n, order m purely imaginary argument $\mathrm{i}\,\mathrm{sh}\,\alpha$. The $P_n^m(\cos\beta)$ is known as the associated Legendre function of real argument $\cos\beta$. The transformation of the unperturbed potential (9) into spheroidal system is:

$$U_0(\alpha,\beta,\varphi) = -B_0 f[c_I\,\mathrm{ch}\,\alpha\sin\beta\cos\varphi + s_I\,\mathrm{sh}\,\alpha\cos\beta]. \tag{23}$$

The dependence on β is given by $\sin\beta$ in the first term and by $\cos\beta$ in the second one so we must take in (22) the degree number $n = 1$ and the dependence on φ will be represented by the order numbers $m = 0, 1$ also in potentials U_1^* and $U_T(\alpha,\beta,\varphi)$. This is guaranteed by the orthogonality of goniometric functions $\cos m\varphi$ and $\sin m\varphi$ on the interval $\varphi \in \langle 0, 2\pi)$. Similarly, the dependence on β in (23) is via $\sin\beta \equiv P_1^1(\cos\beta)$ and by $P_1(\cos\beta) = \cos\beta$. The orthogonality of Legendre functions $P_n^m(\cos\beta)$ implicates this dependence on β in both potential U_1^* and U_T, so we will have degree number $n = 1$. In the theory of the associated spherical functions of purely imaginary argument *(Smythe, 1968)* it is proved that we can calculate the dependence on α by the following functions

$$P_1^0(\mathrm{i}\,\xi) = \mathrm{i}\,\xi, \quad P_1^1(\mathrm{i}\,\xi) = \sqrt{1+\xi^2}, \tag{24}$$

$$Q_1^0(\mathrm{i}\,\xi) = \xi\,\mathrm{arctg}(1/\xi) - 1, \quad Q_1^1(\mathrm{i}\,\xi) = \frac{-\xi}{\sqrt{1+\xi^2}} + \sqrt{1+\xi^2}\,\mathrm{arctg}(1/\xi), \tag{25}$$

where we substituted $\xi = \mathrm{sh}\,\alpha$. It can be found that $P_1^0(\mathrm{i}\,\mathrm{sh}\,\alpha) = \mathrm{i}\,\mathrm{sh}\,\alpha$ and $P_1^1(\mathrm{i}\,\mathrm{sh}\,\alpha) = (1+\mathrm{sh}^2\alpha)^{1/2} = \mathrm{ch}\,\alpha$. These functions are bounded for $\alpha \to 0$, but tend to infinity for $\alpha \to \infty$, so they cannot occur in the perturbing potential U_1^*. The functions of the second kind $Q_1^0(\mathrm{i}\,\xi)$ and $Q_1^1(\mathrm{i}\,\xi)$ are not acceptable for the interior potential $U_T(\alpha,\beta,\varphi)$ because they would produce singular $\mathrm{grad}\,U_T(\alpha,\beta,\varphi)$ for $\alpha \to 0$ as was pointed out by *Lebedev (1963)*. In the book *(Smythe, 1968)* we can also find the more suitable expressions for $Q_1^0(\mathrm{i}\,\xi)$ and $Q_1^1(\mathrm{i}\,\xi)$, namely for $\xi > 1$:

$$Q_1^0(\mathrm{i}\,\xi) = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+3)} \frac{1}{\xi^{2k+2}}, \tag{26}$$

$$Q_1^1(\mathrm{i}\,\xi) = 2\sqrt{1+\xi^2} \sum_{k=0}^{\infty} \frac{(-1)^k(k+1)}{(2k+3)} \frac{1}{\xi^{2k+3}}. \tag{27}$$

It is clear that both these functions have zero limit for $\alpha \to \infty$. In view of discussion above it is clear that α dependence of the interior potential U_T must be given via functions $\mathrm{sh}\,\alpha$ and $\mathrm{ch}\,\alpha$, so this potential will be simple multiple of the exciting potential terms:

$$U_T(\alpha,\beta,\varphi) = -B_0 f[c_I C_2\,\mathrm{ch}\,\alpha \sin\beta\cos\varphi + s_I D_2\,\mathrm{sh}\,\alpha\cos\beta]. \tag{28}$$

The perturbing potential U_1^* outside the spheroid must be dependent on α via functions $Q_1^0(\mathrm{i}\,\mathrm{sh}\,\alpha)$ and $Q_1^1(\mathrm{i}\,\mathrm{sh}\,\alpha)$ and β,φ dependence will be the same as in (28), so it is of the form:

$$U_1^*(\alpha,\beta,\varphi) = -B_0 f[c_I C_1 q_1^1(\mathrm{sh}\,\alpha)\sin\beta\cos\varphi + s_I D_1 q_1(\mathrm{sh}\,\alpha)\cos\beta]. \tag{29}$$

Here we use real form of the functions $Q_1^0(\mathrm{i}\,\mathrm{sh}\,\alpha) \equiv q_1(\mathrm{sh}\,\alpha)$ and $Q_1^1(\mathrm{i}\,\mathrm{sh}\,\alpha) \equiv q_1^1(\mathrm{sh}\,\alpha)$ according to their expressions, because the r.h.s of (25) are real expressions and $U_T(\alpha,\beta,\varphi)$ is also expressed by the real functions. The total potential outside of spheroid is:

$$U_1(\alpha,\beta,\varphi) = U_0 + U_1^* = -B_0 f\left\{c_I\left[\mathrm{ch}\,\alpha + C_1 q_1^1(\mathrm{sh}\,\alpha)\right]\sin\beta\cos\varphi + \right.$$
$$\left. + s_I\left[\mathrm{sh}\,\alpha + D_1 q_1(\mathrm{sh}\,\alpha)\right]\cos\beta\right\} \tag{30}$$

The coefficients C_2, D_2 and C_1, D_1 which determine change of the potentials of the magnetic field from boundary conditions on the surface S of the spheroid where $\alpha = \alpha_0$ whose normal $\boldsymbol{n}$ direction is in the unit vector $\boldsymbol{e}_\alpha$. According to (11) there must be:

$$[U_T]_{\alpha_0} = \mu_r \, [U_1]_{\alpha_0} \,, \tag{31}$$

$$[\partial U_T / \partial \alpha]_{\alpha_0} = [\partial U_1 / \partial \alpha]_{\alpha_0} \,. \tag{32}$$

These boundary conditions must be satisfied for all β and φ and if we use orthogonality of the spherical function $\cos\beta$ (for $m = 0$, $n = 1$) and $\sin\beta\cos\varphi$ (for $m = 1$, $n = 1$) we will obtain four linear equations to determine coefficients $C_2, D_2$and C_1, D_1. For the mode $m = 0$, $n = 1$ we have equations:

$$D_2 \operatorname{sh}\alpha_0 = \mu_r \operatorname{sh}\alpha_0 + \mu_r D_1 q_1(\operatorname{sh}\alpha_0),$$

$$D_2 \operatorname{ch}\alpha_0 = \operatorname{ch}\alpha_0 + D_1 \operatorname{ch}\alpha_0 \, q_1'(\operatorname{sh}\alpha_0),$$

which gives the solution:

$$D_1 = (\mu_r - 1) t_0 \left[t_0 \, q_1'(t_0) - \mu_r q_1(t_0) \right]^{-1}, \tag{33}$$

$$D_2 = 1 + D_1 q_1'(t_0), \tag{34}$$

where $t_0 = \operatorname{sh}\alpha_0$. The orthogonality of the mode $m = 1$, $n = 1$ gives equations:

$$C_2 \operatorname{ch}\alpha_0 = \mu_r \operatorname{ch}\alpha_0 + \mu_r C_1 \, q_1^1(\operatorname{sh}\alpha_0),$$

$$C_2 \operatorname{sh}\alpha_0 = \operatorname{sh}\alpha_0 + C_1 \operatorname{ch}\alpha_0 \, q_1^{1'}(\operatorname{sh}\alpha_0).$$

Putting $t_0 = \operatorname{sh}\alpha_0$ we have $\operatorname{ch}\alpha_0 = \sqrt{1 + t_0^2}$ and we obtain solution:

$$C_1 = (\mu_r - 1) t_0 \sqrt{1 + t_0^2} \left[(t_0^2 + 1) q_1^{1'}(t_0) - \mu_r t_0 q_1^1(t_0) \right]^{-1}, \tag{35}$$

$$C_2 = 1 + t_0^{-1} \sqrt{1 + t_0^2} C_1 \, q_1^{1'}(t_0). \tag{36}$$

In this manner we can calculate the necessary potentials and also their gradients, to obtain $\boldsymbol{B} = -\,\mathrm{grad}\, U$. The formulae (33) and (35) for coefficients D_1 and C_1 have zero values for the non magnetic spheroid ($\mu_r = 1$), which gives zero perturbing potential. In this case we have coefficients D_2 and C_2 equal to 1, which means that $U_T = U_0$.

3. Calculations of the magnetic field components

Now we pay our attention to the calculations of the potential and magnetic field in Carthesian coordinates. The expression (28) of the interior potential can be easily transformed since according to (12) we have $x = f\,\mathrm{ch}\,\alpha \sin\beta \cos\varphi$, so that:

$$U_T(x, y, z) = -B_0[c_I\, C_2\, x + s_I\, D_2\, z]. \tag{37}$$

It corresponds to the uniform magnetic field $\boldsymbol{B}_T \equiv (B_0\, c_I C_2, 0, B_0\, s_I\, D_2)$, in the Carthesian system. The inclination I^* of this magnetic field is different from the angle I, its tangent is clearly

$$\mathrm{tg}\, I^* = (D_2/C_2)\, \mathrm{tg}\, I. \tag{38}$$

The modulus of $\boldsymbol{B}_T$ is changed compared to B_0 via factor

$$F_T = |\boldsymbol{B}_T| \cdot B_0^{-1} = \left[(c_I\, C_2)^2 + (s_I\, D_2)^2\right]^{1/2}. \tag{39}$$

The potential $U_1(\alpha, \beta, \varphi)$ outside the spheroid is the sum of the unperturbed potential $U_0(\alpha, \beta, \varphi)$ and perturbing potential $U_1^*(\alpha, \beta, \varphi)$. This perturbing potential and its gradient is calculated in a network of (x, y, z) variables, so we must calculate proper spheroidal coordinates (α, β, φ). The values of $\mathrm{ch}\,\alpha, \mathrm{sh}\,\alpha$ can be calculated by using the transformation relations (12) and properties of confocal ellipses. We know that the coordinate line $\alpha = \mathrm{const}$ is ellipse with equation (14) in (r, z) plane their foci are in points $r = \pm f$ in the plane $z = 0$, major semiaxis is $f\,\mathrm{ch}\,\alpha$ and minor semiaxis is $f\,\mathrm{sh}\,\alpha$. For every (r, z) point of this ellipse is the sum of distances from the first and second focus equal to the doubled value of major semiaxis which is $2f\,\mathrm{ch}\,\alpha$. There must hold:

$$\left[(r-f)^2+z^2\right]^{1/2}+\left[(r+f)^2+z^2\right]^{1/2}=2f\,\mathrm{ch}\,\alpha, \tag{40}$$

where $r = \sqrt{x^2+y^2}$. From this equation we can determine $\mathrm{ch}\,\alpha$ since $f=\sqrt{a^2-b^2}$ is constant given by the contour ellipse of the spheroid and creates whole family of confocal ellipses $\alpha =$ const. From known value of $\mathrm{ch}\,\alpha$ we can determine $\mathrm{sh}\,\alpha$ by the relation

$$\mathrm{sh}\,\alpha=\left[\mathrm{ch}^2\alpha-1\right]^{1/2} \tag{41}$$

$$\text{and}\quad e^{\alpha}=\mathrm{ch}\,\alpha+\mathrm{sh}\,\alpha. \tag{42}$$

Then we can easily determine also the value of coordinate β, using (12), which gives:

$$\cos\beta=z/(f\,\mathrm{sh}\,\alpha), \tag{43}$$

for $z=0$ and $r>f$ these relation holds also true (there we have $\mathrm{ch}\,\alpha=r/f$ and $\beta=\pi/2$). Inside the focal circle $z=0$, $r<f$ we must be more careful. The value of α is zero and from (34) we have:

$$2f\,\mathrm{ch}\,\alpha=|r-f|+|r+f|=f-r+r+f=2f, \tag{44}$$

so we obtain $\mathrm{ch}\,\alpha=1, \mathrm{sh}\,\alpha=0$. But inside this circle the value of coordinate β is changing as follows from the equation of confocal hyperboloids (19) where we put $z=0$ and then:

$$\sin\beta=r/f. \tag{45}$$

For the azimuthal angle φ there is a simple relation:

$$\mathrm{tg}\,\varphi=y/x. \tag{46}$$

Using these formulae we can assign to each x,y,z point its spheroidal coordinates (α,β,φ) and calculate perturbing potential:

$$U_1^*(\alpha,\beta,\varphi)=-B_0 f\left[c_I C_1 q_1^1(\mathrm{sh}\,\alpha)\sin\beta\cos\varphi+s_I D_1 q_1(\mathrm{sh}\,\alpha)\cos\beta\right] \tag{47}$$

and also components of the anomalous magnetic field outside the spheroid:

$$\boldsymbol{B}^*(\alpha,\beta,\varphi)=-\,\mathrm{grad}\,U_1^*(\alpha,\beta,\varphi),$$

$$B^*_\alpha = -\frac{1}{h_\alpha}\frac{\partial U^*_1}{\partial \alpha}, \quad B^*_\beta = -\frac{1}{h_\beta}\frac{\partial U^*_1}{\partial \beta}, \quad B^*_\varphi = -\frac{1}{h_\varphi}\frac{\partial U^*_1}{\partial \varphi}, \tag{48}$$

where Lame's metrical parameters are given by (21). These derivatives can be easily calculated, but we need to transform these spheroidal vector components into Carthesian ones. We can use the relations given in *Madelung (1957)* (with proper changes of the spheroidal coordinates notation):

$$\begin{aligned} B^*_x &= B^*_r \cos\varphi - B^*_\varphi \sin\varphi, \\ B^*_y &= B^*_r \sin\varphi - B^*_\varphi \cos\varphi, \\ B^*_z &= \left[-B^*_\beta \sin\beta \operatorname{sh}\alpha + B^*_\alpha \operatorname{ch}\alpha\cos\beta\right]\cdot\left[\operatorname{ch}^2\alpha - \sin^2\beta\right]^{-1/2}, \end{aligned} \tag{49}$$

where

$$B^*_r = \left[-B^*_\alpha \sin\beta \operatorname{sh}\alpha + B^*_\beta \operatorname{ch}\alpha\cos\beta\right]\cdot\left[\operatorname{ch}^2\alpha - \sin^2\beta\right]^{-1/2},$$

is the radial magnetic component in x, y plane.

For practical purpose we will calculate the anomalous magnetic field $\boldsymbol{B}^*$ components outside the ellipsoid given by (49). We put the vertical field anomaly ΔZ and total field anomaly ΔT:

$$\Delta Z = B^*_z(x,y,z), \quad \Delta T = B^*_x \cos I \cos A + B^*_y \cos I \sin A + B^*_z \sin I, \tag{50}$$

while we consider azimuth A of the primary field $\boldsymbol{B}_0$ to be zero value, because our x lies in the plane of local magnetic meridian.

4. Numerical calculations and discussion

For the numerical calculations we choose the oblate spheroid with semiaxes $a = 500$ m, $b = 100$ m and susceptibility quite high $\kappa = 0.1$. The inclination of the field $\boldsymbol{B}_0$ we put $I = 75°$. For practical needs it is suitable to perform model calculations of ΔZ and ΔT at some plane $z = z_p$ above the spheroid and also for some model of the hill surface above the magmatic body. The calculated values of $\Delta Z(x, y, z_p)$ and $\Delta Z(x, y, z_p)$ are normalized by the value B_0 and these values multiplied by the factor 1000. Then it is clear that if we put $B_0 = 50000$ nT and the value $(\Delta Z/B_0) \times 1000 = 12$ then the real value $\Delta Z = 600\,\mathrm{nT}$. In Fig. 3a we have plotted also the isolines of perturbing potential $U_1^*(x, y, z_p)$ divided by B_0. For better clarity the curve of U_1^* along the x profile for $y = 0$ is also presented. One can see that the field $U_1^*(x, y, z)$ is very similar to the field of magnetic dipole inclined in the direction I_0. Fig. 3b presents the isolines of $\Delta Z(x, y, z_p)$ together with the profile curve along the x axis, Fig. 3c concerns $\Delta T(x, y, z_p)$. We can see that the maximum of ΔT is shifted to the left of the point ($x = 0$, $y = 0$) clearly due to inclination value $I = 75°$. There exists also a region of negative values at $x/a \approx 1$. If the inclination would be $90°$, the pattern of ΔT will be like concentric circles above the centrum of spheroid. In the table of each figure there are given values of the spheroid parameters, together with value of the changed inclination I^* of $\boldsymbol{B}_T$ inside the spheroid and also the parameter F_T defined by (39). We can see that $I^* = 75.84°$, it differs by small angle $0.84°$ from $I = 75°$. Also the modulus of $\boldsymbol{B}_T$ is greater by the factor $F_T = 1.0823$ in comparison to B_0.

The next set of calculations was performed for the points (x, y, z) distributed on the surface of the hill having the form of the cutted cone above the spheroid. The profile curve of the cone is given in the bottom part of Fig. 4a. The function $H(x, y)$ is calculated by the formula

$$H(x,y) = \begin{cases} h_b & \text{if } r \geq r_b \\ h_b + q(r_b - r) & \text{if } r \in (r_u, r_b), \\ h_u & \text{if } r \in \langle 0, r_u) \end{cases} \tag{51}$$

where $r = \sqrt{x^2 + y^2}$, h_b, h_u are the bottom, upper high of the hill, respectively ($h_b < h_u$) and r_b, r_u are the radii of bottom and upper circle of the hill, while $r_b > r_u$. The slope factor q is clearly $q = (h_u - h_b)/(r_b - r_u)$. The vertical coordinate is set to $z = -H(x, y)$. The height isolines of the hill are drawn in the upper part of Fig. 4a, these are circles. In Figs 4b,c are presented isolines of δZ and ΔT on the hill points. We can see that their values rapidly decrease with the distance from the spheroid, because the upper circle of the hill is in the height $h_u = 750\,\mathrm{m}$. The pattern of isolines of ΔZ and ΔT is more deformed in comparison to the pattern presented in Figs 3b,c for the plane $z_p = -0.5a$.

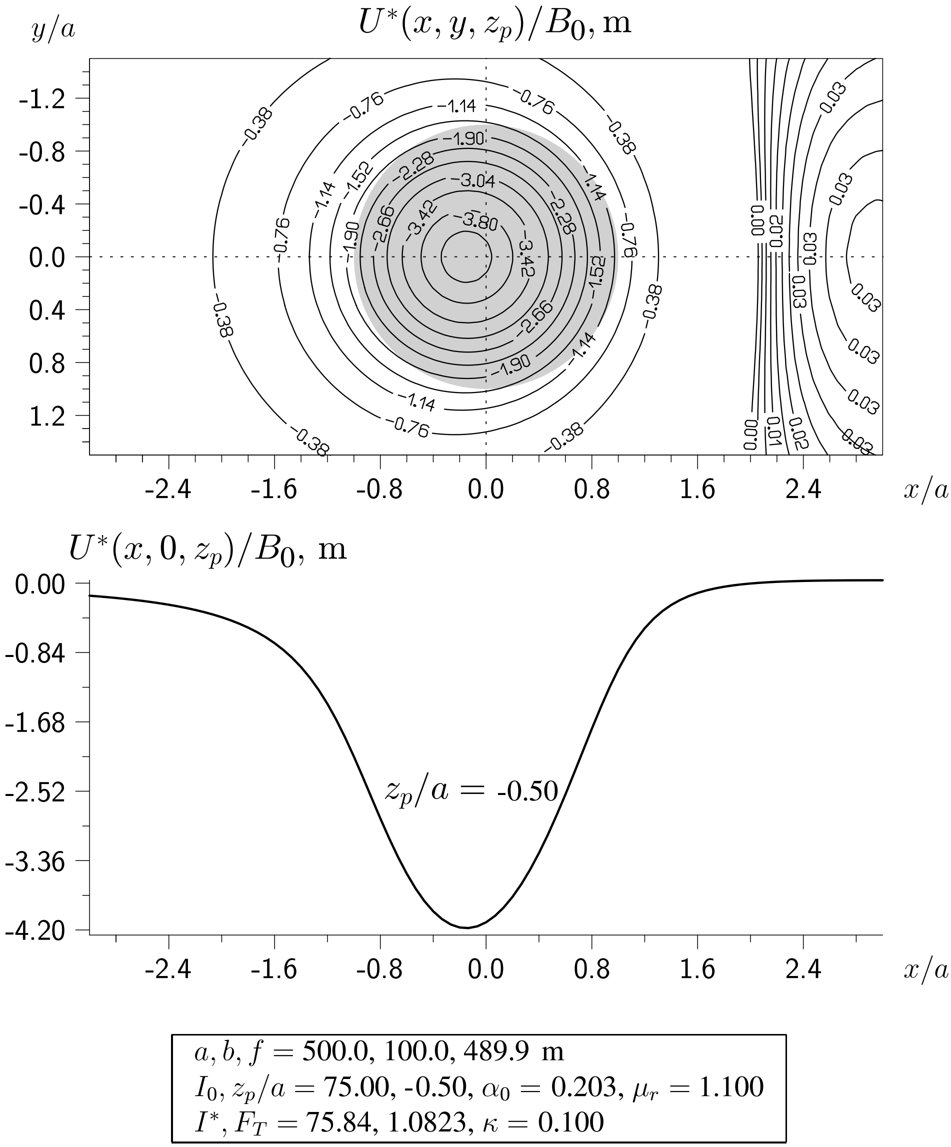
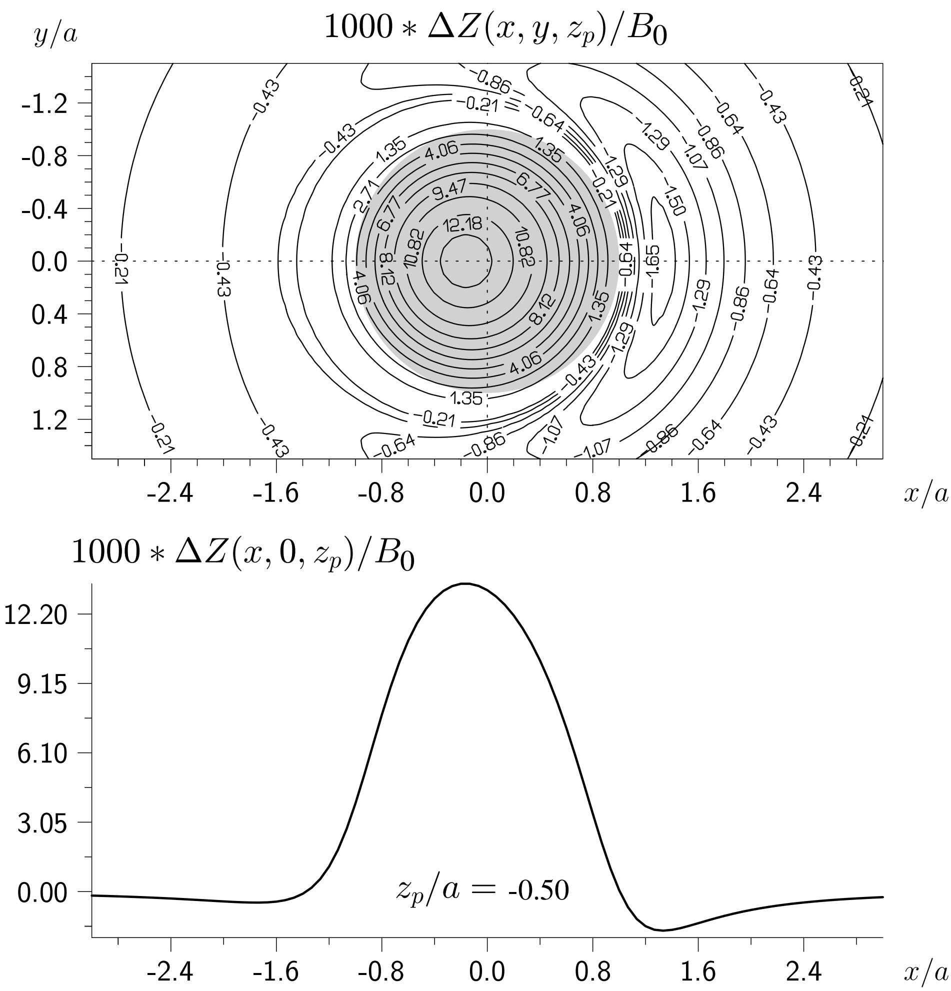
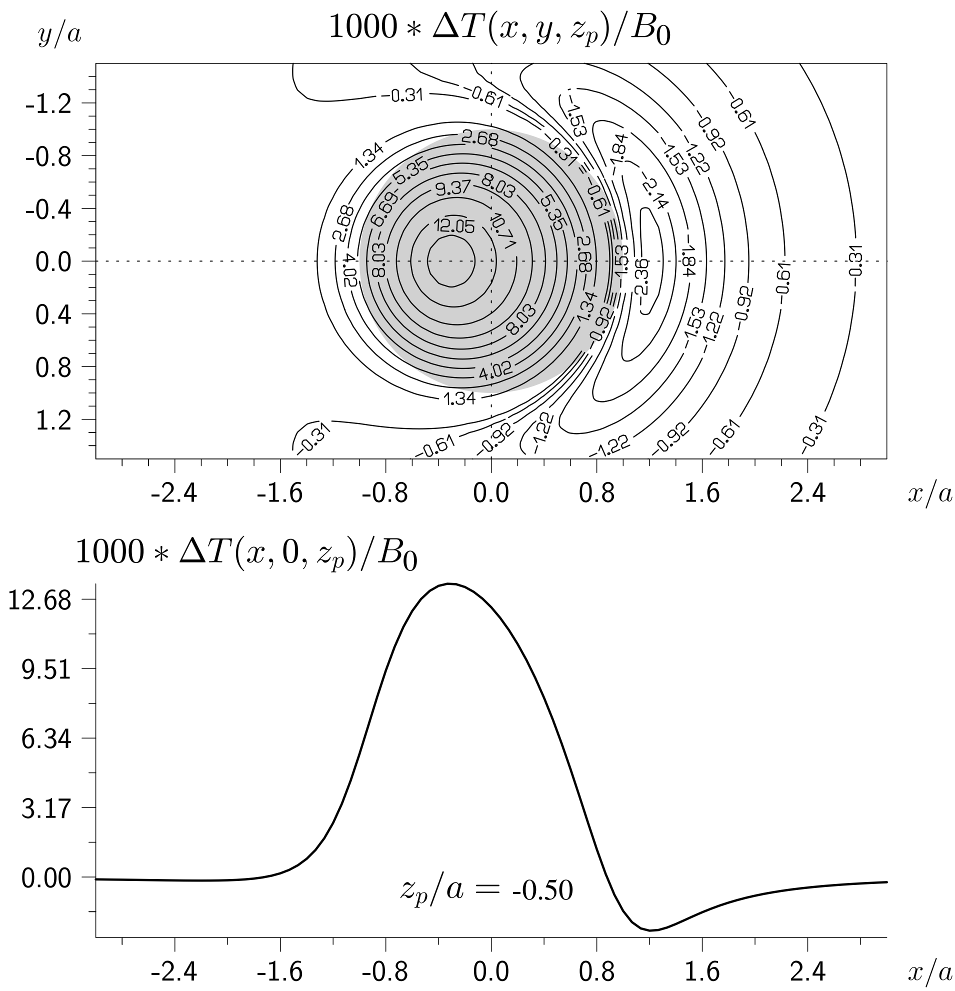


Fig. 3a. Isolines of the anomalous potential above the oblated spheroidal body at the plane $z_p = -0.5a$. The bottom curve presents values along the profile $y = 0$ for the level $z_p/a = -0.5$.



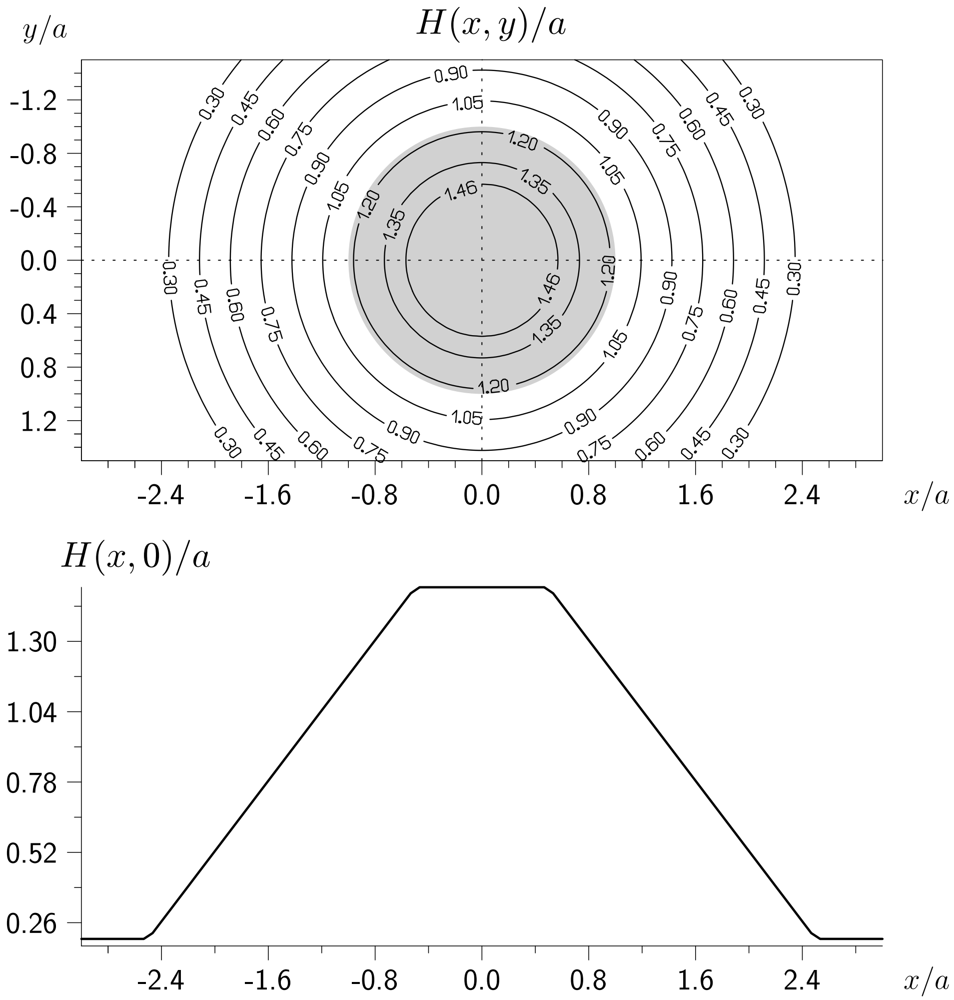
$a, b, f = 500.0, 100.0, 489.9$ m
$I_0, z_p/a = 75.00, -0.50, \alpha_0 = 0.203, \mu_r = 1.100$
$I^*, F_T = 75.84, 1.0823, \kappa = 0.100$

Fig. 3b. The isolines of the relative ΔZ anomaly above the oblated spheroidal body at the level $z_p = -0.5a$. The bottom curve shows the profile values at $y = 0$ for the level $z_p/a = -0.5$.



$a, b, f =$ 500.0, 100.0, 489.9 m
$I_0, z_p/a =$ 75.00, -0.50, $\alpha_0 =$ 0.203, $\mu_r =$ 1.100
$I^*, F_T =$ 75.84, 1.0823, $\kappa =$ 0.100

Fig. 3c. The isolines of the relative ΔT anomaly above the oblated spheroidal body at the level $z_p = -0.5a$. The bottom curve shows the profile values at $y = 0$ for the level $z_p/a = -0.5$.



$a, b, f = 500.0,\ 100.0,\ 489.9$ m
$h_b, h_u, r_b, r_u = 100.0,\ 750.0, 1250.0,\ 250.0$ m
$I_0, \mu_r = 75.00,\ 1.1000,\ \alpha_0 = 0.203$
$I^*, F_T = 75.84,\ 1.0823,\ \kappa_T = 0.100$

Fig. 4a. Isolines of the surface of the cutted cone $H(x, y)$ defined by equation (51) which represents hill above the oblate spheroidal magmatic body. The bottom curve shows hights profile at $y = 0$.

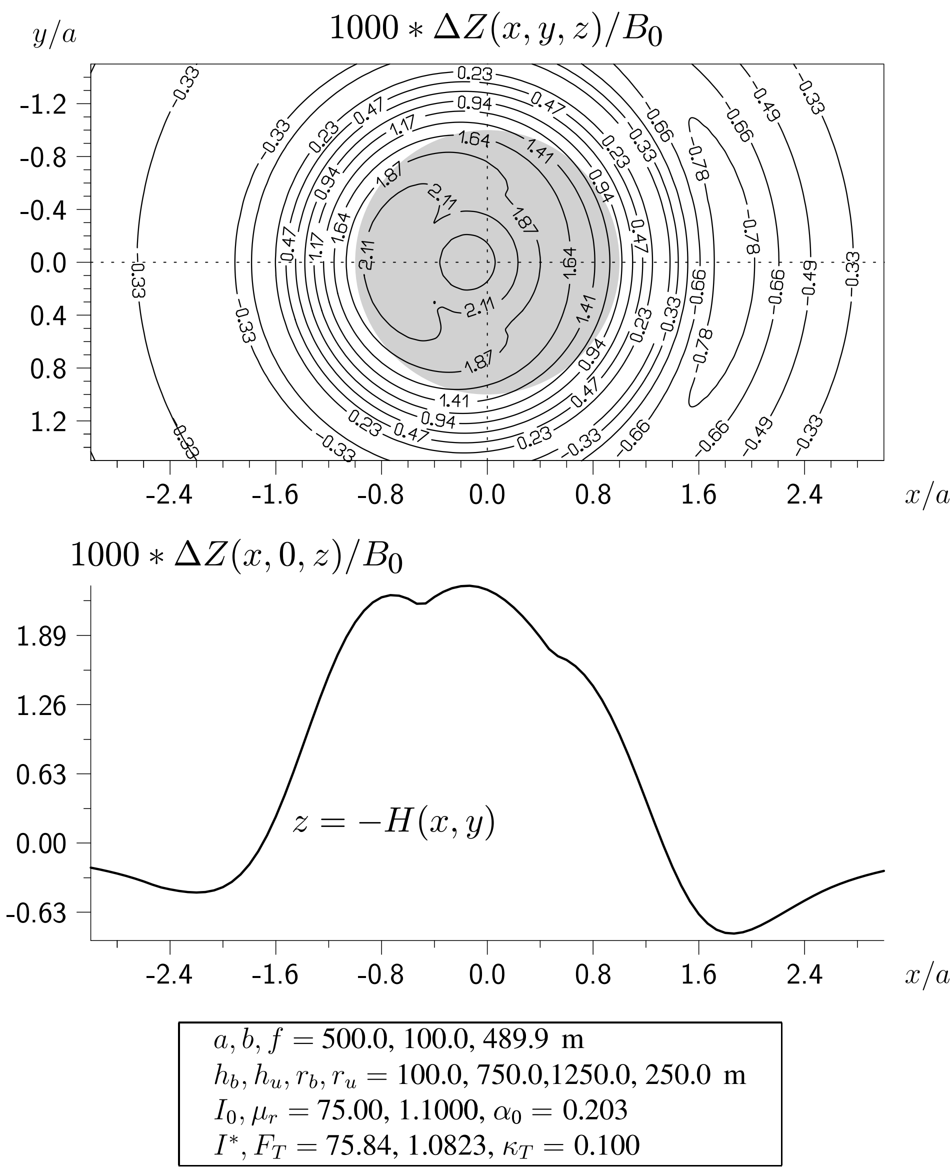
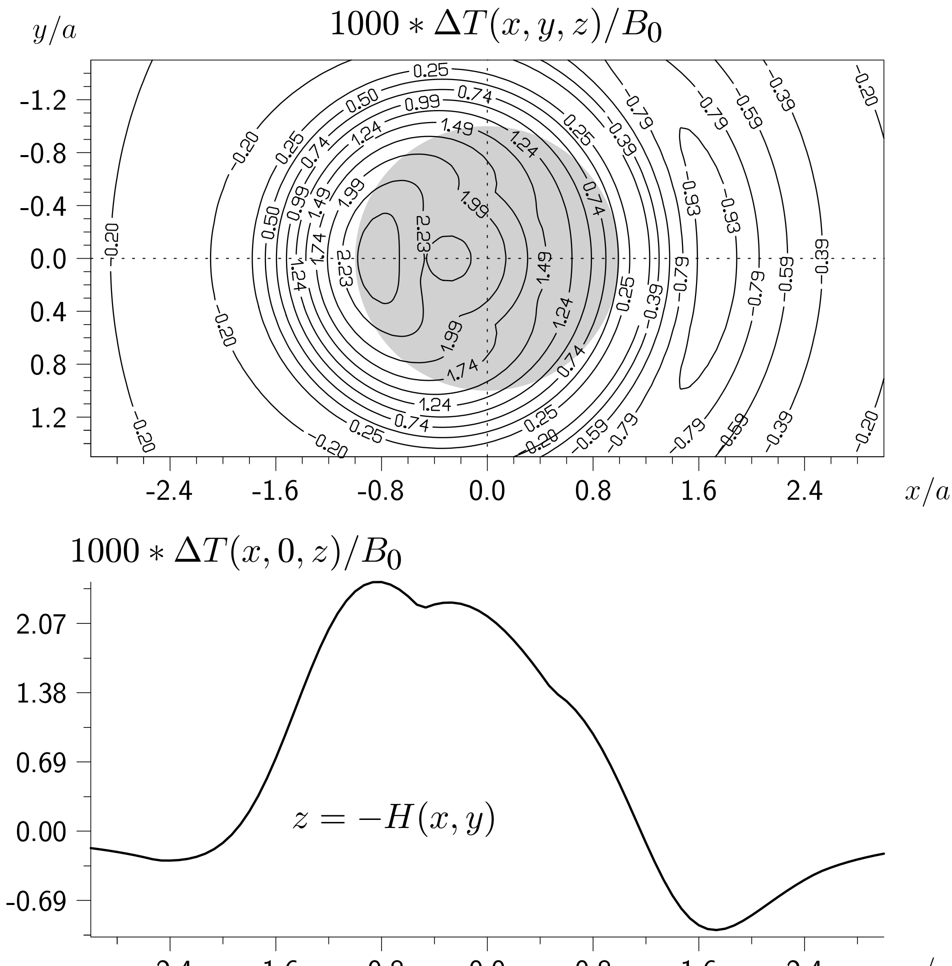


Fig. 4b. The isolines of the relative ΔZ anomaly above the oblate spheroidal body on the surface depicted in Fig. 4a. The bottom curve shows the profile values at $y = 0$ along the hill.



$a, b, f = 500.0,\ 100.0,\ 489.9$ m
$h_b, h_u, r_b, r_u = 100.0,\ 750.0, 1250.0,\ 250.0$ m
$I_0, \mu_r = 75.00,\ 1.1000,\ \alpha_0 = 0.203$
$I^*, F_T = 75.84,\ 1.0823,\ \kappa_T = 0.100$

Fig. 4c. The isolines of the relative ΔT anomaly above the oblate spheroidal body on the surface depicted in Fig. 4a. The bottom curve shows the profile values at $y = 0$ along the hill.

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References

Arfken G., 1966: Mathematical methods for physicists, Academic Press, London.

Hvoždara, M., 2008: Groundwater and geothermal anomalies due to a prolate spheroid. Contrib. Geophys. Geod., **39**, 2, 95–119.

Hvoždara M., Vozár J., 2010: Anomalies of geomagnetic field due to a vertical prolate rotational ellipsoid. Contrib. Geophys. Geod., **40**, 3, 185–205.

Lebedev N. N., 1963: Special functions and their applications. Fizmatgiz, Moscow (in Russian).

Madelung E., 1957: Die Mathematischen Hilfsmittel des Physikers. Spriger-Verlag, Berlin.

Morse P. M., Feshbach H., 1973: Methods of theoretical physics. McGraw-Hill, New York.

Muratov R. Z., 1976: Potentials of ellipsoid. Atomizdat, Moscow, 144 p. (in Russian).

Smythe W. R., 1968: Static and dynamic electricity. McGraw-Hill, New York.

Stratton J. A., 1941: Theory of electromagnetic field. McGraw-Hill, New York.