Gravitational field of the homogeneous rotational ellipsoidal body: a simple derivation and applications

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Abstract: We calculate the gravitational intensity and potential of a homogeneous body with the shape of the rotational ellipsoid. The calculation is performed in ellipsoidal coordinates and uses the properties of harmonic functions expressed as ellipsoidal harmonics. The resulting formulae for the internal and external fields are expressed in ellipsoidal coordinates and (in the case of external field) also in spherical coordinates. The results are used in the calculation of the gravitational field of a layered body whose layer boundaries are rotational ellipsoids with common centre and rotational axis; the density in each layer is constant. The equilibrium of such a layered rotating body is examined: it is found that there is no equilibrium for such a body except the case that the body is homogeneous (thus proving once more the important, but rarely mentioned, fact).

Key words: gravitational intensity, ellipsoidal coordinates, layered body

1. Introduction

The gravitational field of the homogeneous ellipsoidal body was calculated many times in the past (for the history see *Chandrasekhar, 1969*). The usual way is to calculate the gravitational potential by the volume integration over the interior of the body; the integration is performed mostly in rectangular or spherical coordinate system, the use of ellipsoidal coordinates is less common (for example, *Moritz (1990)*, Chapter 5). We present here quite a different approach for calculating the gravitational intensity and potential generated by the homogeneous body with the shape of the rotational ellipsoid. We first transform the volume integrals for intensity and potential into a surface integrals over the surface of the body; then we calculate these surface integrals using the expansion of the function $1/|\boldsymbol{r}' - \boldsymbol{r}|$ into a series

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of ellipsoidal harmonics. Therefore, the derivation of the formulae for the intensity and potential is (in our view) very short and simple; the rather long explaining text in the following Sections is due to a thorough and mathematically correct description of the particular quantities and mathematical operations. Moreover, we present our own proof of the basic formula for the above mentioned expansion of the function $1/|\boldsymbol{r}' - \boldsymbol{r}|$.

Our approach differs from the others in several aspects: first of all, the emphasis is on the intensity (the potential is treated as a secondary quantity); it is interesting that the resulting formulae (and also their derivation) are simpler for the intensity than for the potential. Second, our derivation is performed entirely in ellipsoidal coordinates; these are denoted in such a way that there can be no confusion with spherical coordinates (this may seem absolutely natural, nevertheless, it is not the case in the common literature and the reader must be explicitly noticed to avoid such a confusion). Third, we use (whenever possible) the vector notation – this is very rarely used in geodetic literature. Fourth, we try to use the mathematically correct expressions and reasoning – this is a weak point of many papers and even of textbooks: for example, there are usually omitted the conditions for the existence of derivatives and integrals of particular functions, the conditions for the expressibility of a particular function in the form of a series (respectively, the existence of the sum of a particular series), the conditions of the validity of particular theorems (like the Gauss theorem), the conditions of the validity of the derivation steps, and so on. Especially grave is the confusion of the Legendre functions $P_n^m(z)$ and $Q_n^m(z)$ of the arbitary complex argument z with the Legendre functions $\mathrm{P}_n^m(u)$ and $\mathrm{Q}_n^m(u)$ of the real argument u (which is bounded by $-1 \le u \le 1$ for the function of the first kind and by $u > 1$ for the function of the second kind). The (more) correct approach makes the explaining text slightly longer, but the gain for the reader (especially for the one who will conduct similar mathematical calculations and derivations) is in our opinion much more important.

Last, but not least, the presented method can be generalized in several ways (we mention here only the expression of gravitational field in form of the integrals over the surface of the body) and thus it is very perspective for the further investigations in this topics.

2. Gravitational potential and intensity of a homogeneous body

Let us consider a body whose boundary is a (sufficiently smooth) surface S; the interior of the body will be denoted as D, while the exterior of the body as $D_{\rm ext}$. Let the density of the matter in the body be the constant ρ; then for any point $\boldsymbol{r}$ the gravitational potential of the body $V(\boldsymbol{r})$ is

$$V(\boldsymbol{r}) = -\,\kappa\rho \int_D \mathrm{d}\tau' \frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|}\,, \tag{1}$$

where κ is the gravitational constant and $\mathrm{d}\tau'$ is the volume element at the point $\boldsymbol{r}'$. The intensity of the gravitational field of the body $\boldsymbol{E}(\boldsymbol{r})$ (equal to the acceleration generated by the gravitational field) is then

$$\boldsymbol{E}(\boldsymbol{r}) = -\,\nabla_{\boldsymbol{r}} V(\boldsymbol{r}) = \kappa\rho \int_D \mathrm{d}\tau'\ \nabla_{\boldsymbol{r}} \frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = -\,\kappa\rho \int_D \mathrm{d}\tau'\ \nabla_{\boldsymbol{r}'} \frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|}\,, \tag{2}$$

where $\nabla_{\boldsymbol{r}}$ is the gradient operator (with respect to the radius-vector $\boldsymbol{r}$). We now transform the expressions for the potential and intensity of the gravitational field into surface integrals using the Gauss theorem: if $\boldsymbol{f}(\boldsymbol{r})$ is a vector function with integrable gradient in domain D, then

$$\int_D \mathrm{d}\tau'\ \nabla_{\boldsymbol{r}'} \cdot \boldsymbol{f}(\boldsymbol{r}') = \int_S \mathrm{d}\boldsymbol{\sigma}' \cdot \boldsymbol{f}(\boldsymbol{s}'), \tag{3}$$

where $\mathrm{d}\boldsymbol{\sigma}'$ is the surface element at the point $\boldsymbol{s}'$ on the surface S oriented outwards from the domain D. Accordingly, if $f(\boldsymbol{r})$ is a scalar function with integrable gradient in the domain D, then

$$\int_D \mathrm{d}\tau'\ \nabla_{\boldsymbol{r}'} f(\boldsymbol{r}') = \int_S \mathrm{d}\boldsymbol{\sigma}'\, f(\boldsymbol{s}')\,. \tag{4}$$

For the intensity we get from (2) according to (4)

$$\boldsymbol{E}(\boldsymbol{r}) = -\,\kappa\rho \int_S \mathrm{d}\boldsymbol{\sigma}'\ \frac{1}{|\boldsymbol{s}' - \boldsymbol{r}|}\,; \tag{5}$$

using the identity

$$\nabla_{\boldsymbol{r}'} \cdot \frac{\boldsymbol{r}' - \boldsymbol{r}}{|\boldsymbol{r}' - \boldsymbol{r}|} = \frac{2}{|\boldsymbol{r}' - \boldsymbol{r}|} \tag{6}$$

we obtain from (1) according to (3) for the potential the expression

$$V(\boldsymbol{r}) = -\frac{1}{2}\,\kappa\rho \int_S \mathrm{d}\boldsymbol{\sigma}' \cdot \frac{\boldsymbol{s}' - \boldsymbol{r}}{|\boldsymbol{s}' - \boldsymbol{r}|}\,. \tag{7}$$

We note that the formulae (5) and (7) are valid for any position of the calculation point $\boldsymbol{r}$ with respect to the body.

3. Ellipsoidal coordinates

We now take into account that the body has the shape of the rotational ellipsoid whose equatorial radius is a and whose excentricity is ε (where $0 \le \varepsilon < 1$); the polar radius of the body is thus $b = a\sqrt{1-\varepsilon^2}$. Therefore it will be advantageous to use for the expression of the gravitational potential and intensity the ellipsoidal coordinates: we follow here the notation and conventions introduced in author's works *Pohánka (1995)* and *Pohánka (1999)* (these will be cited shortly as P95 and P99); for the mathematical background we shall refer mostly to *Bateman and Erdélyi (1953)* (or shortly BE).

Let us first consider the rectangular coordinate system with the origin in the centre of the body and the base vectors **i**, **j**, **k**, where the vector **k** is parallel to the rotational axis of the body. The radius-vector $\boldsymbol{r}$ of an arbitrary point which is expressed in the spherical coordinates r, ϑ, φ as

$$\boldsymbol{r} = r\,\Big(\sin\vartheta\,(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi) + \mathbf{k}\cos\vartheta\Big), \tag{8}$$

will be expressed in the ellipsoidal coordinates (coordinates of oblate spheroid) υ, ξ, ψ as

$$\boldsymbol{r} = a\,\Big(\sqrt{(1-\varepsilon^2)\upsilon^2+\varepsilon^2}\,\sin\xi\,\left(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi\right) + \mathbf{k}\sqrt{1-\varepsilon^2}\,\upsilon\cos\xi\Big), \tag{9}$$

where $\upsilon \ge 0$, $0 \le \xi \le \pi$, $0 \le \psi < 2\pi$ (see BE, 16.1.3, *Heiskanen and Moritz (1967)*, Paragraph 1-19, P95, Section 6, P99, Section 1). These coordinates are defined in such a way that the surface of the body S is given by the condition $\upsilon = 1$ and the interior (exterior) domain D (D_{ext}) determined by the surface S is given by the condition $0 \le \upsilon < 1$ ($\upsilon > 1$). The parametrical expression of the radius-vector $\boldsymbol{s}$ of the point at the surface S is thus

$$\boldsymbol{s}(\xi,\psi) = a\left(\sin\xi\,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi) + \mathbf{k}\sqrt{1-\varepsilon^2}\,\cos\xi\right). \tag{10}$$

The transformation from the ellipsoidal to spherical coordinates is given by the formulae

$$\begin{aligned} r &= a\sqrt{(1-\varepsilon^2)\,\upsilon^2 + \varepsilon^2\sin^2\xi}\,,\\ \vartheta &= \arccos\frac{\sqrt{1-\varepsilon^2}\,\upsilon\cos\xi}{\sqrt{(1-\varepsilon^2)\,\upsilon^2+\varepsilon^2\sin^2\xi}}\,, \quad \varphi = \psi\,; \end{aligned} \tag{11}$$

for the inverse transformation we first introduce the radius of the focal circle e and the Jacobi radial ellipsoidal coordinate s

$$e = a\varepsilon\,, \quad s = a\sqrt{1-\varepsilon^2}\,\upsilon\,, \tag{12}$$

and the function

$$s(\boldsymbol{r},e) = \sqrt{\frac{1}{2}\left(r^2 - e^2 + \sqrt{(r^2-e^2)^2 + 4e^2r^2\cos^2\vartheta}\right)}\,, \tag{13}$$

and then we have

$$\begin{aligned} s &= s(\boldsymbol{r},e)\,, \quad \upsilon = \frac{s}{a\sqrt{1-\varepsilon^2}}\,,\\ \xi &= \arccos\frac{r\cos\vartheta}{s}\,, \quad \psi = \varphi\,. \end{aligned} \tag{14}$$

The vector surface element of the surface S can be calculated by the formula (see P95, Section 2)

$$\mathrm{d}\boldsymbol{\sigma} = \eta\,\partial_\xi\boldsymbol{s}(\xi,\psi)\times\partial_\psi\boldsymbol{s}(\xi,\psi)\,\mathrm{d}\xi\,\mathrm{d}\psi\,, \tag{15}$$

where η ($\eta^2 = 1$) is chosen so that the element has the required orientation. According to (10) we get that $\eta = 1$ and

$$\mathrm{d}\boldsymbol{\sigma} = a^2\,\boldsymbol{o}(\xi,\psi)\,\mathrm{d}\Xi\,, \tag{16}$$

where

$$\boldsymbol{o}(\xi,\psi) = \sqrt{1-\varepsilon^2}\,\sin\xi\,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi) + \mathbf{k}\cos\xi \tag{17}$$

and

$$\mathrm{d}\Xi = \sin\xi \, \mathrm{d}\xi \, \mathrm{d}\psi \,. \tag{18}$$

We finally introduce the unit vector

$$\boldsymbol{v} = \sin\xi \,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi) + \mathbf{k}\cos\xi \tag{19}$$

and from now on we can write any function $f(\xi,\psi)$ briefly as $f(\boldsymbol{v})$. The unit vector $\boldsymbol{n}(\boldsymbol{s})$ of the external normal to the surface S at the point $\boldsymbol{s}$ is then given by

$$\boldsymbol{n}(\boldsymbol{s}) = \boldsymbol{n}(\boldsymbol{s}(\boldsymbol{v})) = \frac{\boldsymbol{o}(\boldsymbol{v})}{k(\boldsymbol{v})}\,, \tag{20}$$

where

$$k(\boldsymbol{v}) = |\boldsymbol{o}(\boldsymbol{v})| = \sqrt{1-\varepsilon^2\sin^2\!\xi}\,. \tag{21}$$

Following the conventions adopted in P95, Section 1, for any function $f(\boldsymbol{r})$ we denote as $[f(\boldsymbol{s})]_{\mathrm{int}}$ ($[f(\boldsymbol{s})]_{\mathrm{ext}}$) the inner (outer) limit (with respect to the domain D) of this function at the point $\boldsymbol{s}$ on the surface S. The gradient of the function $f(\boldsymbol{r})$ is in the ellipsoidal coordinates equal to

$$\nabla_{\boldsymbol{r}} f(\boldsymbol{r}) = (\nabla_{\boldsymbol{r}}\upsilon)\,\partial_\upsilon f(\boldsymbol{r}) + (\nabla_{\boldsymbol{r}}\xi)\,\partial_\xi f(\boldsymbol{r}) + (\nabla_{\boldsymbol{r}}\psi)\,\partial_\psi f(\boldsymbol{r})\,; \tag{22}$$

in order to calculate the quantities $\boldsymbol{o}(\boldsymbol{v})\cdot[\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\mathrm{int}}$ and $\boldsymbol{o}(\boldsymbol{v})\cdot[\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\mathrm{ext}}$ we have to know the scalar products $\boldsymbol{o}(\boldsymbol{v})\cdot[\nabla_{\boldsymbol{r}}\upsilon]_{\boldsymbol{r}=\boldsymbol{s}}$, $\boldsymbol{o}(\boldsymbol{v})\cdot[\nabla_{\boldsymbol{r}}\xi]_{\boldsymbol{r}=\boldsymbol{s}}$, $\boldsymbol{o}(\boldsymbol{v})\cdot[\nabla_{\boldsymbol{r}}\psi]_{\boldsymbol{r}=\boldsymbol{s}}$. Following P99, Section 1, we use the fact that $\nabla_{\boldsymbol{r}}\boldsymbol{r}$ is the identity tensor and we have

$$\nabla_{\boldsymbol{r}}\boldsymbol{r} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk} = (\nabla_{\boldsymbol{r}}\upsilon)\,\partial_\upsilon \boldsymbol{r} + (\nabla_{\boldsymbol{r}}\xi)\,\partial_\xi \boldsymbol{r} + (\nabla_{\boldsymbol{r}}\psi)\,\partial_\psi \boldsymbol{r}\,. \tag{23}$$

Using the expression (9) we can easily show that vectors $\partial_\upsilon\boldsymbol{r}$, $\partial_\xi\boldsymbol{r}$, $\partial_\psi\boldsymbol{r}$ are mutually orthogonal (this expresses the orthogonality of the ellipsoidal coordinate system); thus we have

$$\partial_\upsilon\boldsymbol{r} = (\nabla_{\boldsymbol{r}}\upsilon)\,(\partial_\upsilon\boldsymbol{r})^2, \quad \partial_\xi\boldsymbol{r} = (\nabla_{\boldsymbol{r}}\xi)\,(\partial_\xi\boldsymbol{r})^2, \quad \partial_\psi\boldsymbol{r} = (\nabla_{\boldsymbol{r}}\psi)\,(\partial_\psi\boldsymbol{r})^2\,, \tag{24}$$

and this implies that vectors $\nabla_{\boldsymbol{r}}\upsilon$, $\nabla_{\boldsymbol{r}}\xi$, $\nabla_{\boldsymbol{r}}\psi$ are also mutually orthogonal. From (9) and (17) we get

$$[\partial_\upsilon\boldsymbol{r}]_{\upsilon=1} = a\sqrt{1-\varepsilon^2}\,\boldsymbol{o}(\boldsymbol{v}) \tag{25}$$

and from the first formula in (24) and (21) we obtain

$$[\nabla_{\boldsymbol{r}} \upsilon]_{\boldsymbol{r}=\boldsymbol{s}} = \frac{\boldsymbol{o}(\boldsymbol{v})}{a\sqrt{1-\varepsilon^2}\, k(\boldsymbol{v})^2}\,; \tag{26}$$

the vectors $[\nabla_{\boldsymbol{r}} \xi]_{\boldsymbol{r}=\boldsymbol{s}}$ and $[\nabla_{\boldsymbol{r}} \psi]_{\boldsymbol{r}=\boldsymbol{s}}$ are orthogonal to the vector $\boldsymbol{o}(\boldsymbol{v})$. Therefore we finally have

$$\boldsymbol{o}(\boldsymbol{v}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\mathrm{int}} = \frac{1}{a\sqrt{1-\varepsilon^2}}\, \lim\nolimits_{\upsilon\to 1-} \partial_\upsilon f(\boldsymbol{r})\,, \tag{27}$$

$$\boldsymbol{o}(\boldsymbol{v}) \cdot [\nabla_{\boldsymbol{s}} f(\boldsymbol{s})]_{\mathrm{ext}} = \frac{1}{a\sqrt{1-\varepsilon^2}}\, \lim\nolimits_{\upsilon\to 1+} \partial_\upsilon f(\boldsymbol{r})\,. \tag{28}$$

4. Harmonic functions in ellipsoidal coordinates

In order to express functions defined on the surface S we introduce the spherical functions $\mathrm{Y}_{n,m}(\boldsymbol{v})$: we follow here the definition presented in P95, Section 3. These functions are defined on the surface of the (abstract) unit sphere by the formula (for the connection between the vector $\boldsymbol{v}$ and coordinates ξ, ψ see (19))

$$\mathrm{Y}_{n,m}(\boldsymbol{v}) = C_{n,m}\, \mathrm{P}_n^{|m|}(\cos\xi)\, \mathrm{e}^{\mathrm{i}m\psi}\,, \tag{29}$$

where $\mathrm{P}_n^m(u)$ is the associated Legendre function of real argument u, $|u| \le 1$ (for definition see BE, Chapter 3) and $C_{n,m}$ is the normalization coefficient: for $|m| \le n$ it reads

$$C_{n,m} = \sqrt{(2n+1)\, \frac{(n-|m|)!}{(n+|m|)!}}\,, \tag{30}$$

otherwise $C_{n,m} = 0$. Spherical functions are complex (it holds true that $\mathrm{Y}^*_{n,m}(\boldsymbol{v}) = \mathrm{Y}_{n,-m}(\boldsymbol{v})$, where asterisk denotes the complex conjugation) and they form an orthonormal system: for $|m| \le n$, $|m'| \le n'$ it holds true that

$$\frac{1}{4\pi} \int \mathrm{d}\Xi\ \mathrm{Y}^*_{n,m}(\boldsymbol{v})\ \mathrm{Y}_{n',m'}(\boldsymbol{v}) = \delta_{n,n'}\, \delta_{m,m'}\,, \tag{31}$$

where the angular element $\mathrm{d}\Xi$ is given by (18).

Spherical functions form a complete system of functions on the unit sphere (note that vector $\boldsymbol{v}$ has unit length): any continuous function $f(\boldsymbol{v})$ can be expressed as a convergent series

$$f(\boldsymbol{v}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} f_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v}), \tag{32}$$

where the coefficients $f_{n,m}$ are given by

$$f_{n,m} = \frac{1}{4\pi} \int \mathrm{d}\Xi\; f(\boldsymbol{v})\, \mathrm{Y}^*_{n,m}(\boldsymbol{v}); \tag{33}$$

it is evident that for a real function $f(\boldsymbol{v})$ it holds true that $f^*_{n,m} = f_{n,-m}$. For the criterion of convergence of the series (32) in terms of coefficients $f_{n,m}$ see P95, Section 3.

The adopted abbreviate notation of sums is the following: $\sum_{n\geq k}$ ($\sum_{n\leq k}$) is the summation over n (the first variable in the condition) from k to ∞ (from $-\infty$ to k) and $\sum_{k\leq n\leq l}$ is the summation over n (the middle variable in the condition) from k to l if $k \leq l$, and zero otherwise ($\sum_{|n|\leq k}$ is the abbreviation of $\sum_{-k\leq n\leq k}$).

Now we can turn to the harmonic functions: as the Laplace equation can be solved in ellipsoidal coordinates by separation of variables, harmonic functions can be expressed as a series of ellipsoidal harmonics (see for example *Heiskanen and Moritz (1967)*, Paragraph 1-20, or *Hobson (1931)*, Paragraph 252). These series contain the associated Legendre functions $P_n^m(z)$ and $Q_n^m(z)$ for purely imaginary argument $z = \mathrm{i}u$, $u \geq 0$ (for definitions of these functions for arbitrary complex z see BE, Chapter 3). In order to remove the imaginary argument we introduce here (following P99, Section 3) the modified Legendre functions $\mathrm{p}_n^m(u)$ and $\mathrm{q}_n^m(u)$ defined by

$$P_n^m(\mathrm{i}u) = \mathrm{i}^n\, \mathrm{p}_n^m(u), \tag{34}$$

$$Q_n^m(\mathrm{i}u) = \frac{(-1)^m}{\mathrm{i}^{n+1}}\, \mathrm{q}_n^m(u). \tag{35}$$

We shall need only the functions $\mathrm{p}_n^{|m|}(u)$ and $\mathrm{q}_n^{|m|}(u)$, $|m| \leq n$, $u \geq 0$; they can be expressed in the form of integrals (see BE, 3.7.14, 3.7.12)

$$\mathrm{p}_n^{|m|}(u) = \frac{(n+|m|)!}{n!}\, \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}t\; (u + \sqrt{u^2+1}\, \cos t)^n \cos mt, \tag{36}$$

$$\mathrm{q}_n^{|m|}(u) = \frac{n!}{(n-|m|)!} \int_0^\infty \mathrm{d}t \, \frac{\operatorname{ch} mt}{(u + \sqrt{u^2+1} \operatorname{ch} t)^{n+1}} \,. \tag{37}$$

For the properties of these modified Legendre functions see P99, Section 3; we note here only the fact that (for $u \geq 0$) the function $\mathrm{q}_n^{|m|}(u)$ is always positive and the function $\mathrm{p}_n^{|m|}(u)$ is positive with the exception of the case that $u = 0$ and $n - |m|$ is odd (then it is equal to zero). From the Wronskian formula for the Legendre functions (see BE, 3.2.13, 1.2.15, P99, Section 5) we get using (34) and (35) the formula

$$\partial\mathrm{p}_n^{|m|}(u)\,\mathrm{q}_n^{|m|}(u) - \mathrm{p}_n^{|m|}(u)\,\partial\mathrm{q}_n^{|m|}(u) = \frac{(n+|m|)!}{(n-|m|)!}\,\frac{1}{u^2+1}\,, \tag{38}$$

where we have denoted

$$\partial\mathrm{p}_n^m(u) = \partial_u \mathrm{p}_n^m(u)\,, \quad \partial\mathrm{q}_n^m(u) = \partial_u \mathrm{q}_n^m(u)\,. \tag{39}$$

Now we are able to express harmonic functions as a series of ellipsoidal harmonics. We first define the quantity

$$\kappa(\varepsilon) = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\,; \tag{40}$$

then any function $F(\boldsymbol{r})$ harmonic in the domain D can be expressed for $\boldsymbol{r} \in D$ (thus $0 \leq \upsilon < 1$) in the form

$$F(\boldsymbol{r}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} F_{n,m}\, \frac{\mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))}\, \mathrm{Y}_{n,m}(\boldsymbol{v})\,, \tag{41}$$

and any function $G(\boldsymbol{r})$ harmonic in the domain D_{ext} and tending to zero for $|\boldsymbol{r}| \to \infty$ can be expressed for $\boldsymbol{r} \in D_{\mathrm{ext}}$ (thus $\upsilon > 1$) in the form

$$G(\boldsymbol{r}) = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} G_{n,m}\, \frac{\mathrm{q}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))}\, \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{42}$$

(for the connection between vectors $\boldsymbol{r}$, $\boldsymbol{v}$ and coordinates υ, ξ, ψ see (9) and (19)). At the surface S we then have the limiting values

$$[F(\boldsymbol{s})]_{\mathrm{int}} = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} F_{n,m}\, \mathrm{Y}_{n,m}(\boldsymbol{v})\,, \tag{43}$$

$$[G(\boldsymbol{s})]_{\text{ext}} = \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} G_{n,m} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \, ; \tag{44}$$

for the limiting values of the derivative with respect to υ we obtain

$$\lim\nolimits_{\upsilon\to 1-} \partial_\upsilon F(\boldsymbol{r}) = \kappa(\varepsilon) \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} F_{n,m} \, \frac{\partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \, , \tag{45}$$

$$\lim\nolimits_{\upsilon\to 1+} \partial_\upsilon G(\boldsymbol{r}) = \kappa(\varepsilon) \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} G_{n,m} \, \frac{\partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \, . \tag{46}$$

According to (27), (28) and (40) we then have

$$\boldsymbol{o}(\boldsymbol{v}) \cdot [\nabla_{\boldsymbol{s}} F(\boldsymbol{s})]_{\text{int}} = \frac{1}{a\,\varepsilon} \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} F_{n,m} \, \frac{\partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{p}_n^{|m|}(\kappa(\varepsilon))} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \, , \tag{47}$$

$$\boldsymbol{o}(\boldsymbol{v}) \cdot [\nabla_{\boldsymbol{s}} G(\boldsymbol{s})]_{\text{ext}} = \frac{1}{a\,\varepsilon} \sum\nolimits_{n\geq 0} \sum\nolimits_{|m|\leq n} G_{n,m} \, \frac{\partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon))}{\mathrm{q}_n^{|m|}(\kappa(\varepsilon))} \, \mathrm{Y}_{n,m}(\boldsymbol{v}) \, . \tag{48}$$

Now we write the third Green identity for the functions $F(\boldsymbol{r})$ and $G(\boldsymbol{r})$: for any point $\boldsymbol{r} \in D$ we have

$$\int_S \mathrm{d}\boldsymbol{\sigma}' \cdot \left([F(\boldsymbol{s}')]_{\text{int}} \left[\nabla_{\boldsymbol{s}'} \frac{1}{|\boldsymbol{s}' - \boldsymbol{r}|} \right]_{\text{int}} - [\nabla_{\boldsymbol{s}'} F(\boldsymbol{s}')]_{\text{int}} \, \frac{1}{|\boldsymbol{s}' - \boldsymbol{r}|} \right) = = -\,4\pi\, F(\boldsymbol{r}) \, , \tag{49}$$

and for any point $\boldsymbol{r} \in D_{\text{ext}}$ we have

$$\int_S \mathrm{d}\boldsymbol{\sigma}' \cdot \left([G(\boldsymbol{s}')]_{\text{ext}} \left[\nabla_{\boldsymbol{s}'} \frac{1}{|\boldsymbol{s}' - \boldsymbol{r}|} \right]_{\text{ext}} - [\nabla_{\boldsymbol{s}'} G(\boldsymbol{s}')]_{\text{ext}} \, \frac{1}{|\boldsymbol{s}' - \boldsymbol{r}|} \right) = = 4\pi\, G(\boldsymbol{r}) \, . \tag{50}$$

The function $1/|\boldsymbol{r}' - \boldsymbol{r}|$ is harmonic with respect to both variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ everywhere with the exception of the case $\boldsymbol{r} = \boldsymbol{r}'$. In order to express this function as a series of ellipsoidal harmonics, we introduce for any $\upsilon_0 > 0$ the ellipsoidal surface $S(\upsilon_0)$ defined by the condition $\upsilon = \upsilon_0$; this surface is the boundary of domains $D_{\text{int}}(\upsilon_0)$ and $D_{\text{ext}}(\upsilon_0)$, defined by the conditions

$0 \leq \upsilon < \upsilon_0$ and $\upsilon > \upsilon_0$, respectively. Then it is evident that the function $1/|\boldsymbol{r}' - \boldsymbol{r}|$ is harmonic with respect to both variables $\boldsymbol{r}$ and $\boldsymbol{r}'$ if one of them is in the domain $D_{\mathrm{int}}(\upsilon_0)$ and the other one is in the domain $D_{\mathrm{ext}}(\upsilon_0)$.

We first consider the case that $0 < \upsilon_0 < 1$ and $\boldsymbol{r} \in D_{\mathrm{int}}(\upsilon_0)$, $\boldsymbol{r}' \in D_{\mathrm{ext}}(\upsilon_0)$, and we write the function $1/|\boldsymbol{r}' - \boldsymbol{r}|$ as a series of ellipsoidal harmonics of the variable $\boldsymbol{r}'$ (note that this function is tending to zero for $|\boldsymbol{r}'| \to \infty$):

$$\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = \sum\nolimits_{n' \geq 0} \sum\nolimits_{|m'| \leq n'} I^{-}_{n',m'}(\upsilon_0, \boldsymbol{r})\, \mathrm{q}_{n'}^{|m'|}(\kappa(\varepsilon)\,\upsilon')\; \mathrm{Y}^{*}_{n',m'}(\boldsymbol{v}')\,. \tag{51}$$

As for any point $\boldsymbol{s}' \in S$ it holds true that $\boldsymbol{s}' \in D_{\mathrm{ext}}(\upsilon_0)$, we can insert the expression (51) in the formula (49); we choose the function $F(\boldsymbol{r})$ in the form $\mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)\; \mathrm{Y}_{n,m}(\boldsymbol{v})$ and using (16), (27), (40) and (47) we obtain

$$\frac{1}{4\pi} \int \mathrm{d}\Xi'\; \mathrm{Y}_{n,m}(\boldsymbol{v}') \sum\nolimits_{n' \geq 0} \sum\nolimits_{|m'| \leq n'} I^{-}_{n',m'}(\upsilon_0, \boldsymbol{r}) \;\cdot$$
$$\cdot\, \frac{a}{\varepsilon} \Big(\mathrm{p}_n^{|m|}(\kappa(\varepsilon))\; \partial \mathrm{q}_{n'}^{|m'|}(\kappa(\varepsilon)) - \partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\; \mathrm{q}_{n'}^{|m'|}(\kappa(\varepsilon)) \Big)\, \mathrm{Y}^{*}_{n',m'}(\boldsymbol{v}') =$$
$$= -\, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)\; \mathrm{Y}_{n,m}(\boldsymbol{v}) \tag{52}$$

(here the dots at the end of the first line and at the beginning of the second line do not denote the scalar product, but a simple multiplication; they appear because the particular term is too long to be written in a single line). According to (31) we get

$$I^{-}_{n,m}(\upsilon_0, \boldsymbol{r})\, \frac{a}{\varepsilon} \Big(\mathrm{p}_n^{|m|}(\kappa(\varepsilon))\; \partial \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) - \partial \mathrm{p}_n^{|m|}(\kappa(\varepsilon))\; \mathrm{q}_n^{|m|}(\kappa(\varepsilon)) \Big) =$$
$$= -\, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)\; \mathrm{Y}_{n,m}(\boldsymbol{v})\,, \tag{53}$$

and using (38) we obtain

$$I^{-}_{n,m}(\upsilon_0, \boldsymbol{r}) = \frac{1}{a\varepsilon}\, \frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon)\; \mathrm{Y}_{n,m}(\boldsymbol{v})\,; \tag{54}$$

inserting in (51) we finally get the formula (valid for $0 \leq \upsilon < \upsilon_0 < \upsilon'$, $0 < \upsilon_0 < 1$)

$$\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = \frac{1}{a\varepsilon} \sum\nolimits_{n \geq 0} \sum\nolimits_{|m| \leq n} \frac{(n-|m|)!}{(n+|m|)!}\, \mathrm{q}_n^{|m|}(\kappa(\varepsilon)\,\upsilon')\; \mathrm{p}_n^{|m|}(\kappa(\varepsilon)\,\upsilon) \;\cdot$$
$$\cdot\, \mathrm{Y}^{*}_{n,m}(\boldsymbol{v}')\; \mathrm{Y}_{n,m}(\boldsymbol{v})\,. \tag{55}$$

Now we consider the case that $\upsilon_0 > 1$ and $\boldsymbol{r} \in D_{\text{ext}}(\upsilon_0)$, $\boldsymbol{r}' \in D_{\text{int}}(\upsilon_0)$, and we write the function $1/|\boldsymbol{r}' - \boldsymbol{r}|$ as a series of ellipsoidal harmonics of the variable $\boldsymbol{r}'$:

$$\frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = \sum_{n'\geq 0} \sum_{|m'|\leq n'} I^+_{n',m'}(\upsilon_0, \boldsymbol{r})\, \mathrm{p}^{|m'|}_{n'}(\kappa(\varepsilon)\upsilon')\ \mathrm{Y}^*_{n',m'}(\boldsymbol{\upsilon}')\,. \tag{56}$$

As for any point $\boldsymbol{s}' \in S$ it holds true that $\boldsymbol{s}' \in D_{\text{int}}(\upsilon_0)$, we can insert the expression (56) in the formula (50); we choose the function $G(\boldsymbol{r})$ in the form $\mathrm{q}^{|m|}_n(\kappa(\varepsilon)\upsilon)\ \mathrm{Y}_{n,m}(\boldsymbol{\upsilon})$ and using (16), (28), (40) and (48) we obtain

$$\begin{aligned} \frac{1}{4\pi}\int \mathrm{d}\Xi'\ \mathrm{Y}_{n,m}(\boldsymbol{\upsilon}') \sum_{n'\geq 0} \sum_{|m'|\leq n'} I^+_{n',m'}(\upsilon_0, \boldsymbol{r}) \cdot \\ \cdot \frac{a}{\varepsilon}\Big(\mathrm{q}^{|m|}_n(\kappa(\varepsilon))\ \partial\mathrm{p}^{|m'|}_{n'}(\kappa(\varepsilon)) - \partial\mathrm{q}^{|m|}_n(\kappa(\varepsilon))\ \mathrm{p}^{|m'|}_{n'}(\kappa(\varepsilon))\Big)\,\mathrm{Y}^*_{n',m'}(\boldsymbol{\upsilon}') = \\ = \mathrm{q}^{|m|}_n(\kappa(\varepsilon)\upsilon)\ \mathrm{Y}_{n,m}(\boldsymbol{\upsilon})\,. \end{aligned} \tag{57}$$

According to (31) we get

$$\begin{aligned} I^+_{n,m}(\upsilon_0, \boldsymbol{r})\,\frac{a}{\varepsilon}\Big(\mathrm{q}^{|m|}_n(\kappa(\varepsilon))\ \partial\mathrm{p}^{|m|}_n(\kappa(\varepsilon)) - \partial\mathrm{q}^{|m|}_n(\kappa(\varepsilon))\ \mathrm{p}^{|m|}_n(\kappa(\varepsilon))\Big) = \\ = \mathrm{q}^{|m|}_n(\kappa(\varepsilon)\upsilon)\ \mathrm{Y}_{n,m}(\boldsymbol{\upsilon})\,, \end{aligned} \tag{58}$$

and using (38) we obtain

$$I^+_{n,m}(\upsilon_0, \boldsymbol{r}) = \frac{1}{a\varepsilon}\,\frac{(n-|m|)!}{(n+|m|)!}\,\mathrm{q}^{|m|}_n(\kappa(\varepsilon)\upsilon)\ \mathrm{Y}_{n,m}(\boldsymbol{\upsilon})\,; \tag{59}$$

inserting in (56) we finally get the formula (valid for $0 \leq \upsilon' < \upsilon_0 < \upsilon$, $\upsilon_0 > 1$)

$$\begin{aligned} \frac{1}{|\boldsymbol{r}' - \boldsymbol{r}|} = \frac{1}{a\varepsilon} \sum_{n\geq 0} \sum_{|m|\leq n} \frac{(n-|m|)!}{(n+|m|)!}\,\mathrm{p}^{|m|}_n(\kappa(\varepsilon)\upsilon')\ \mathrm{q}^{|m|}_n(\kappa(\varepsilon)\upsilon) \cdot \\ \cdot \mathrm{Y}^*_{n,m}(\boldsymbol{\upsilon}')\ \mathrm{Y}_{n,m}(\boldsymbol{\upsilon})\,. \end{aligned} \tag{60}$$

Note that the expressions on the r.h.s. of the formulae (55) and (60) do not contain the parameter υ_0; as the function $1/|\boldsymbol{r}' - \boldsymbol{r}|$ is symmetric with respect to variables $\boldsymbol{r}$ and $\boldsymbol{r}'$, the formula (55) holds true for $0 \leq \upsilon < \upsilon'$ and the formula (60) holds true for $0 \leq \upsilon' < \upsilon$.

We have thus proved the formula for the expansion of the function $1/|\boldsymbol{r}'-\boldsymbol{r}|$ as a series of ellipsoidal harmonics presented and proved in *Hobson (1931)*, Paragraph 251 (it was used in P99, Section 5, without proof); we note that our proof is different and much shorter. Using the formulae (31), (55) and (60) we immediately obtain for $\boldsymbol{r}\in D$ (thus $0 \le \upsilon < 1$) the formula

$$\frac{1}{4\pi}\int \mathrm{d}\Xi' \, \frac{\mathrm{Y}_{n,m}(\boldsymbol{v}')}{|\boldsymbol{s}(\boldsymbol{v}')-\boldsymbol{r}|} = \frac{1}{a\varepsilon}\,\frac{(n-|m|)!}{(n+|m|)!}\,\mathrm{q}_n^{|m|}(\kappa(\varepsilon))\;\mathrm{p}_n^{|m|}(\kappa(\varepsilon)\upsilon)\;\mathrm{Y}_{n,m}(\boldsymbol{v})\,, \quad (61)$$

and for $\boldsymbol{r}\in D_{\mathrm{ext}}$ (thus $\upsilon > 1$) the formula

$$\frac{1}{4\pi}\int \mathrm{d}\Xi' \, \frac{\mathrm{Y}_{n,m}(\boldsymbol{v}')}{|\boldsymbol{s}(\boldsymbol{v}')-\boldsymbol{r}|} = \frac{1}{a\varepsilon}\,\frac{(n-|m|)!}{(n+|m|)!}\,\mathrm{p}_n^{|m|}(\kappa(\varepsilon))\;\mathrm{q}_n^{|m|}(\kappa(\varepsilon)\upsilon)\;\mathrm{Y}_{n,m}(\boldsymbol{v})\,. \quad (62)$$

5. Expressions for the gravitational potential and intensity of a homogeneous ellipsoidal body

We are now able to present the formulae for the gravitational potential and intensity of a homogeneous ellipsoidal body: we shall need the formulae (5), (7), (10), (16), (17), (61) and (62). The formula (5) now reads according to (16)

$$\boldsymbol{E}(\boldsymbol{r}) = -\,\kappa\rho a^2 \int \mathrm{d}\Xi' \, \frac{\boldsymbol{o}(\boldsymbol{v}')}{|\boldsymbol{s}(\boldsymbol{v}')-\boldsymbol{r}|}\,; \quad (63)$$

for the potential we first write using (5) and (7)

$$V(\boldsymbol{r}) = \frac{1}{2}\Big(U(\boldsymbol{r}) - \boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r})\Big)\,, \quad (64)$$

where

$$U(\boldsymbol{r}) = -\,\kappa\rho \int_S \mathrm{d}\boldsymbol{\sigma}'\cdot\frac{\boldsymbol{s}'}{|\boldsymbol{s}'-\boldsymbol{r}|}\,, \quad (65)$$

and according to (10), (16) and (17) we obtain

$$U(\boldsymbol{r}) = -\,\kappa\rho a^2 \int \mathrm{d}\Xi' \, \frac{\boldsymbol{o}(\boldsymbol{v}')\cdot\boldsymbol{s}(\boldsymbol{v}')}{|\boldsymbol{s}(\boldsymbol{v}')-\boldsymbol{r}|} = -\,\kappa\rho a^3\sqrt{1-\varepsilon^2}\int \mathrm{d}\Xi' \, \frac{1}{|\boldsymbol{s}(\boldsymbol{v}')-\boldsymbol{r}|}\,. \quad (66)$$

From the definition (29) we obtain that $\mathrm{Y}_{0,0}(\boldsymbol{v}) = 1$, the quantity $\cos\xi$ is a multiple of $\mathrm{Y}_{1,0}(\boldsymbol{v})$ and the quantity $\sin\xi\,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi)$ is a linear combination of $\mathrm{Y}_{1,-1}(\boldsymbol{v})$ and $\mathrm{Y}_{1,1}(\boldsymbol{v})$. Then we easily get from (61) for $\boldsymbol{r}\in D$ the formulae

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\,\frac{1}{|\boldsymbol{s}(\boldsymbol{v}') - \boldsymbol{r}|} = \frac{1}{a\varepsilon}\,\mathrm{q}_0^0(\kappa(\varepsilon))\;\mathrm{p}_0^0(\kappa(\varepsilon)\,\upsilon)\,, \tag{67}$$

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\,\frac{\cos\xi'}{|\boldsymbol{s}(\boldsymbol{v}') - \boldsymbol{r}|} = \frac{1}{a\varepsilon}\,\mathrm{q}_1^0(\kappa(\varepsilon))\;\mathrm{p}_1^0(\kappa(\varepsilon)\,\upsilon)\,\cos\xi\,, \tag{68}$$

$$\begin{aligned}\frac{1}{4\pi}\int \mathrm{d}\Xi'\,\frac{\sin\xi'\,(\mathbf{i}\cos\psi' + \mathbf{j}\sin\psi')}{|\boldsymbol{s}(\boldsymbol{v}') - \boldsymbol{r}|} &= \\ &= \frac{1}{2a\varepsilon}\,\mathrm{q}_1^1(\kappa(\varepsilon))\;\mathrm{p}_1^1(\kappa(\varepsilon)\,\upsilon)\,\sin\xi\,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi)\,,\end{aligned} \tag{69}$$

and from (62) for $\boldsymbol{r}\in D_{\mathrm{ext}}$ the formulae

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\,\frac{1}{|\boldsymbol{s}(\boldsymbol{v}') - \boldsymbol{r}|} = \frac{1}{a\varepsilon}\,\mathrm{p}_0^0(\kappa(\varepsilon))\;\mathrm{q}_0^0(\kappa(\varepsilon)\,\upsilon)\,, \tag{70}$$

$$\frac{1}{4\pi}\int \mathrm{d}\Xi'\,\frac{\cos\xi'}{|\boldsymbol{s}(\boldsymbol{v}') - \boldsymbol{r}|} = \frac{1}{a\varepsilon}\,\mathrm{p}_1^0(\kappa(\varepsilon))\;\mathrm{q}_1^0(\kappa(\varepsilon)\,\upsilon)\,\cos\xi\,, \tag{71}$$

$$\begin{aligned}\frac{1}{4\pi}\int \mathrm{d}\Xi'\,\frac{\sin\xi'\,(\mathbf{i}\cos\psi' + \mathbf{j}\sin\psi')}{|\boldsymbol{s}(\boldsymbol{v}') - \boldsymbol{r}|} &= \\ &= \frac{1}{2a\varepsilon}\,\mathrm{p}_1^1(\kappa(\varepsilon))\;\mathrm{q}_1^1(\kappa(\varepsilon)\,\upsilon)\,\sin\xi\,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi)\,.\end{aligned} \tag{72}$$

According to (36) we have

$$\mathrm{p}_0^0(u) = 1\,, \tag{73}$$

$$\mathrm{p}_1^0(u) = u\,, \quad \mathrm{p}_1^1(u) = \sqrt{u^2+1}\,, \tag{74}$$

$$\mathrm{p}_2^0(u) = \frac{1}{2}\,(3u^2+1)\,, \tag{75}$$

and from the formula (37) we obtain

$$\mathrm{q}_n^0(u) = \int_0^\infty \mathrm{d}t \, \frac{1}{(u + \sqrt{u^2+1}\,\mathrm{ch}\,t)^{n+1}} \,, \tag{76}$$

$$\mathrm{q}_n^1(u) = n \int_0^\infty \mathrm{d}t \, \frac{\mathrm{ch}\,t}{(u + \sqrt{u^2+1}\,\mathrm{ch}\,t)^{n+1}} \,. \tag{77}$$

Taking the partial derivative of both sides of (76) (see (39)) and rearranging the terms in numerator under the integral sign we easily obtain the formula

$$\mathrm{q}_{n+1}^0(u) = -\,u\,\mathrm{q}_n^0(u) - \frac{u^2+1}{n+1}\,\partial\mathrm{q}_n^0(u)\,; \tag{78}$$

rearranging the numerator under the integral sign in (77) and using (76) we obtain the formula

$$\mathrm{q}_n^1(u) = \frac{n}{\sqrt{u^2+1}} \Big(\mathrm{q}_{n-1}^0(u) - u\,\mathrm{q}_n^0(u) \Big) \,. \tag{79}$$

The expression for the function $\mathrm{q}_0^0(u)$ can be easily obtained by calculating the integral in (76) for $n = 0$

$$\mathrm{q}_0^0(u) = \frac{\mathrm{i}}{2}\,\ln\frac{u-\mathrm{i}}{u+\mathrm{i}} = \mathrm{arctg}\,\frac{1}{u}\,, \tag{80}$$

(alternatively, one can use (35) and BE, 3.6.2, (20)) and thus

$$\partial\mathrm{q}_0^0(u) = -\,\frac{1}{u^2+1}\,. \tag{81}$$

From (78), (79) and (81) we then get

$$\mathrm{q}_1^0(u) = 1 - u\,\mathrm{q}_0^0(u)\,, \tag{82}$$

$$\mathrm{q}_1^1(u) = \frac{1}{\sqrt{u^2+1}} \Big((u^2+1)\,\mathrm{q}_0^0(u) - u \Big) \,, \tag{83}$$

$$\mathrm{q}_2^0(u) = \frac{1}{2} \Big((3\,u^2+1)\,\mathrm{q}_0^0(u) - 3\,u \Big) \,. \tag{84}$$

In order to express explicitly the asymptotic behaviour of these functions for $u \to \infty$ we write them in the form

$$\mathrm{q}_0^0(u) = \frac{1}{u}\,\mathrm{r}_0^0(u)\,, \tag{85}$$

$$\mathrm{q}_1^0(u) = \frac{1}{3\,u^2}\,\mathrm{r}_1^0(u)\,, \quad \mathrm{q}_1^1(u) = \frac{2}{3\,u\sqrt{u^2+1}}\,\mathrm{r}_1^1(u)\,, \tag{86}$$

$$\mathrm{q}_2^0(u) = \frac{2}{15\,u^3}\,\mathrm{r}_2^0(u)\,, \tag{87}$$

and the functions $\mathrm{r}_0^0(u)$, $\mathrm{r}_1^0(u)$, $\mathrm{r}_1^1(u)$ and $\mathrm{r}_2^0(u)$ are given by the formulae

$$\mathrm{r}_0^0(u) = u\ \mathrm{arctg}\,\frac{1}{u}\,, \tag{88}$$

$$\mathrm{r}_1^0(u) = 3\,u^2\left(1 - \mathrm{r}_0^0(u)\right), \tag{89}$$

$$\mathrm{r}_1^1(u) = \frac{3}{2}\left((u^2+1)\,\mathrm{r}_0^0(u) - u^2\right), \tag{90}$$

$$\mathrm{r}_2^0(u) = \frac{15\,u^2}{4}\left((3\,u^2+1)\,\mathrm{r}_0^0(u) - 3\,u^2\right). \tag{91}$$

We also note that there are the following relations between these functions:

$$\mathrm{r}_1^1(u) = \frac{1}{2}\left(3\,\mathrm{r}_0^0(u) - \mathrm{r}_1^0(u)\right), \tag{92}$$

$$\mathrm{r}_2^0(u) = \frac{15\,u^2}{4}\left(\mathrm{r}_0^0(u) - \mathrm{r}_1^0(u)\right). \tag{93}$$

Using the well known expansion of the arctangent function we can write the functions $\mathrm{r}_0^0(u)$, $\mathrm{r}_1^0(u)$, $\mathrm{r}_1^1(u)$ and $\mathrm{r}_2^0(u)$ in the form of series:

$$\mathrm{r}_0^0(u) = \sum\nolimits_{k\geq 0} \frac{(-1)^k}{(2k+1)\,u^{2k}}\,, \tag{94}$$

$$\mathrm{r}_1^0(u) = \sum\nolimits_{k\geq 0} \frac{3\,(-1)^k}{(2k+3)\,u^{2k}}\,, \tag{95}$$

$$\mathrm{r}_1^1(u) = \sum\nolimits_{k\geq 0} \frac{3\,(-1)^k}{(2k+1)(2k+3)\,u^{2k}}\,, \tag{96}$$

$$\mathrm{r}_2^0(u) = \sum\nolimits_{k\geq 0} \frac{15\,(-1)^k\,(k+1)}{(2k+3)(2k+5)\,u^{2k}}\,, \tag{97}$$

which shows that the limit of all of these functions for $u \to \infty$ is equal to 1.

Now we can present the explicit formulae for the gravitational potential and intensity: from (63) we obtain using (17), (68), (69), (71), (72), (74) and (86) for $\boldsymbol{r} \in D$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{4\pi}{3}\kappa\rho a\left(\mathrm{r}_1^1(\kappa(\varepsilon))\sqrt{(1-\varepsilon^2)\upsilon^2+\varepsilon^2}\,\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+ + \mathbf{k}\frac{1}{\sqrt{1-\varepsilon^2}}\,\mathrm{r}_1^0(\kappa(\varepsilon))\,\upsilon\cos\xi\right) \tag{98}$$

and for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{4\pi}{3}\kappa\rho a\frac{1}{\upsilon^2}\left(\frac{\upsilon}{\sqrt{(1-\varepsilon^2)\upsilon^2+\varepsilon^2}}\,\mathrm{r}_1^1(\kappa(\varepsilon)\upsilon)\cdot \cdot\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+\mathbf{k}\frac{1}{\sqrt{1-\varepsilon^2}}\,\mathrm{r}_1^0(\kappa(\varepsilon)\upsilon)\cos\xi\right). \tag{99}$$

Similarly, from (66) we obtain using (67), (70), (73) and (85) for $\boldsymbol{r} \in D$

$$U(\boldsymbol{r}) = -4\pi\kappa\rho a^2\,\mathrm{r}_0^0(\kappa(\varepsilon)) \tag{100}$$

and for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$U(\boldsymbol{r}) = -4\pi\kappa\rho a^2\frac{1}{\upsilon}\,\mathrm{r}_0^0(\kappa(\varepsilon)\upsilon)\,. \tag{101}$$

After introducing the mass of the body

$$M = \frac{4\pi}{3}\rho a^3\sqrt{1-\varepsilon^2}, \tag{102}$$

we obtain from (98) for $\boldsymbol{r} \in D$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{\kappa M}{a^2(1-\varepsilon^2)}\left(\sqrt{1-\varepsilon^2}\,\mathrm{r}_1^1(\kappa(\varepsilon))\sqrt{(1-\varepsilon^2)\upsilon^2+\varepsilon^2}\,\cdot \cdot\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+\mathbf{k}\,\mathrm{r}_1^0(\kappa(\varepsilon))\,\upsilon\cos\xi\right) \tag{103}$$

and from (99) for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{\kappa M}{a^2(1-\varepsilon^2)\upsilon^2}\left(\frac{\sqrt{1-\varepsilon^2}\,\upsilon}{\sqrt{(1-\varepsilon^2)\upsilon^2+\varepsilon^2}}\,\mathrm{r}_1^1(\kappa(\varepsilon)\upsilon)\cdot \cdot\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+\mathbf{k}\,\mathrm{r}_1^0(\kappa(\varepsilon)\upsilon)\cos\xi\right). \tag{104}$$

Similarly, we obtain from (100) for $\boldsymbol{r} \in D$

$$U(\boldsymbol{r}) = -\frac{3\,\kappa M}{a\sqrt{1-\varepsilon^2}}\ \mathrm{r}_0^0(\kappa(\varepsilon)) \tag{105}$$

and from (101) for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$U(\boldsymbol{r}) = -\frac{3\,\kappa M}{a\sqrt{1-\varepsilon^2}\,\upsilon}\ \mathrm{r}_0^0(\kappa(\varepsilon)\,\upsilon)\,. \tag{106}$$

Now we can calculate the quantity $\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r})$: using (9) we obtain from (103) for $\boldsymbol{r} \in D$

$$\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}) = -\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\Big(\mathrm{r}_1^1(\kappa(\varepsilon))\,\Big((1-\varepsilon^2)\,\upsilon^2+\varepsilon^2\Big)\sin^2\!\xi + {} + \mathrm{r}_1^0(\kappa(\varepsilon))\,\upsilon^2\cos^2\!\xi\Big) \tag{107}$$

and from (104) for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}) = -\frac{\kappa M}{a\sqrt{1-\varepsilon^2}\,\upsilon}\Big(\mathrm{r}_1^1(\kappa(\varepsilon)\,\upsilon)\,\sin^2\!\xi + \mathrm{r}_1^0(\kappa(\varepsilon)\,\upsilon)\,\cos^2\!\xi\Big)\,. \tag{108}$$

We rewrite these expressions using the well known definition of the Legendre polynomial $\mathrm{P}_2(\cos\xi)$

$$\mathrm{P}_2(\cos\xi) = \frac{1}{2}\Big(3\,\cos^2\!\xi - 1\Big)\,; \tag{109}$$

for the modification of the formula (107) we still introduce the function $B(\boldsymbol{r})$ (see P99, Section 2; this function was there denoted as $E(\boldsymbol{r})$)

$$B(\boldsymbol{r}) = 1 - \frac{\boldsymbol{r}^2 - (\mathbf{k}\cdot\boldsymbol{r})^2}{a^2} - \frac{(\mathbf{k}\cdot\boldsymbol{r})^2}{a^2\,(1-\varepsilon^2)}\,, \tag{110}$$

which can be expressed in ellipsoidal coordinates according to (9) as

$$B(\boldsymbol{r}) = (1-\upsilon^2)\Big(1-\varepsilon^2\sin^2\!\xi\Big) \tag{111}$$

(therefore we shall write this function also in the form $B(\upsilon,\xi)$). The function $B(\boldsymbol{r})$ is biharmonic because

$$\Delta_{\boldsymbol{r}} B(\boldsymbol{r}) = -\frac{2\,(3-2\varepsilon^2)}{a^2\,(1-\varepsilon^2)} \tag{112}$$

and it vanishes at the surface S

$$B(\boldsymbol{s}) = 0\,. \tag{113}$$

Using the expressions (109) and (111) we can easily calculate that

$$\Big((1-\varepsilon^2)\,\upsilon^2+\varepsilon^2\Big)\,\sin^2\!\xi = \frac{2}{3} - \frac{2}{3}\,\frac{3\,(1-\varepsilon^2)\,\upsilon^2+\varepsilon^2}{3-2\varepsilon^2}\,\mathrm{P}_2(\cos\xi) - \\ - \frac{2\,(1-\varepsilon^2)}{3-2\varepsilon^2}\,B(\upsilon,\xi)\,, \tag{114}$$

$$\upsilon^2\cos^2\!\xi = \frac{1}{3} + \frac{2}{3}\,\frac{3\,(1-\varepsilon^2)\,\upsilon^2+\varepsilon^2}{3-2\varepsilon^2}\,\mathrm{P}_2(\cos\xi) - \frac{1}{3-2\varepsilon^2}\,B(\upsilon,\xi)\,; \tag{115}$$

note that according to (40) and (75) we have

$$\frac{3\,(1-\varepsilon^2)\,\upsilon^2+\varepsilon^2}{3-2\varepsilon^2} = \frac{\mathrm{p}_2^0(\kappa(\varepsilon)\,\upsilon)}{\mathrm{p}_2^0(\kappa(\varepsilon))}\,, \tag{116}$$

and thus according to (41), for both formulae (114) and (115), the first two terms in the expression on the r.h.s. are harmonic in D, while the third term is biharmonic. Inserting the expressions (114), (115) and (116) in (107) and using the formulae (89), (92) and (93) we obtain for $\boldsymbol{r}\in D$

$$\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}) = -\,\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\Bigg(\mathrm{r}_0^0(\kappa(\varepsilon)) - \\ - \frac{4\varepsilon^2}{15\,(1-\varepsilon^2)}\,\mathrm{r}_2^0(\kappa(\varepsilon))\,\frac{\mathrm{p}_2^0(\kappa(\varepsilon)\,\upsilon)}{\mathrm{p}_2^0(\kappa(\varepsilon))}\,\mathrm{P}_2(\cos\xi) - \frac{3\,(1-\varepsilon^2)}{3-2\varepsilon^2}\,B(\upsilon,\xi)\Bigg)\,; \tag{117}$$

similarly, inserting the expressions (114), (115) for $\upsilon = 1$ in (108) and using the formulae (92) and (93) we obtain for $\boldsymbol{r}\in D_{\mathrm{ext}}$

$$\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{r}) = -\,\frac{\kappa M}{a\sqrt{1-\varepsilon^2}\,\upsilon}\Bigg(\mathrm{r}_0^0(\kappa(\varepsilon)\,\upsilon) - \\ - \frac{4\varepsilon^2}{15\,(1-\varepsilon^2)\,\upsilon^2}\,\mathrm{r}_2^0(\kappa(\varepsilon)\,\upsilon)\,\mathrm{P}_2(\cos\xi)\Bigg)\,. \tag{118}$$

The potential $V(\boldsymbol{r})$ is then given by the formula (64): using (105) and (117) we obtain for $\boldsymbol{r}\in D$

$$V(\boldsymbol{r}) = -\,\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\Bigg(\mathrm{r}_0^0(\kappa(\varepsilon)) + \frac{2\varepsilon^2}{15\,(1-\varepsilon^2)}\,\mathrm{r}_2^0(\kappa(\varepsilon))\,\frac{\mathrm{p}_2^0(\kappa(\varepsilon)\,\upsilon)}{\mathrm{p}_2^0(\kappa(\varepsilon))}\,\mathrm{P}_2(\cos\xi) + \\ + \frac{3\,(1-\varepsilon^2)}{2\,(3-2\varepsilon^2)}\,B(\upsilon,\xi)\Bigg)\,; \tag{119}$$

and using (106) and (118) we obtain for $\boldsymbol{r} \in D_{\mathrm{ext}}$

$$V(\boldsymbol{r}) = -\frac{\kappa M}{a\sqrt{1-\varepsilon^2}\,\upsilon}\Big(\mathrm{r}_0^0(\kappa(\varepsilon)\,\upsilon) + \\ + \frac{2\,\varepsilon^2}{15\,(1-\varepsilon^2)\,\upsilon^2}\,\mathrm{r}_2^0(\kappa(\varepsilon)\,\upsilon)\,\mathrm{P}_2(\cos\xi)\Big)\,. \tag{120}$$

We see that the potential $V(\boldsymbol{r})$ is expressed in the domain D as a sum of two harmonic terms and a biharmonic term: according to (102) and (112) it satisfies there the Poisson equation

$$\Delta_{\boldsymbol{r}} V(\boldsymbol{r}) = 4\pi\,\kappa\,\rho\,; \tag{121}$$

in the domain D_{ext} the potential $V(\boldsymbol{r})$ is expressed as a sum of two harmonic terms (this follows from the formulae (42), (85) and (87)).

We present also the formulae for the gravitational potential and intensity at the surface S: these are obtained by taking the limits of their corresponding expressions in D and D_{ext} to the surface S (of course, the interior and exterior limits are identical). From (103) and (104) we obtain

$$\boldsymbol{E}(\boldsymbol{s}) = -\frac{\kappa M}{a^2\,(1-\varepsilon^2)}\Big(\sqrt{1-\varepsilon^2}\,\mathrm{r}_1^1(\kappa(\varepsilon))\,\sin\xi\,(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi) + \\ + \mathbf{k}\,\mathrm{r}_1^0(\kappa(\varepsilon))\,\cos\xi\Big)\,, \tag{122}$$

while from (119) and (120) we obtain using (113)

$$V(\boldsymbol{s}) = -\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\left(\mathrm{r}_0^0(\kappa(\varepsilon)) + \frac{2\,\varepsilon^2}{15\,(1-\varepsilon^2)}\,\mathrm{r}_2^0(\kappa(\varepsilon))\,\mathrm{P}_2(\cos\xi)\right). \tag{123}$$

Finally we express the gravitational potential and intensity outside the body in spherical coordinates as a series of spherical harmonics. The potential $V(\boldsymbol{r})$ and intensity $\boldsymbol{E}(\boldsymbol{r})$ are harmonic functions in domain D_{ext}; taking into account the symmetries of the potential we can express it in the form

$$V(\boldsymbol{r}) = \sum\nolimits_{n\ge 0} V_n\,\frac{a^{2n}}{r^{2n+1}}\,\mathrm{P}_{2n}(\cos\vartheta)\,, \tag{124}$$

where V_n are unknown coefficients. Taking the gradient of the potential we obtain after straightforward calculation

$$\boldsymbol{E}(\boldsymbol{r}) = \sum\nolimits_{n\ge 0} V_n\,\frac{a^{2n}}{r^{2n+2}}\Big(K_n^1(\cos\vartheta)\,\sin\vartheta\,(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi) + \\ + \mathbf{k}\,K_n^0(\cos\vartheta)\Big)\,, \tag{125}$$

where

$$K_n^1(\cos\vartheta) = \cos\vartheta\ \partial\mathrm{P}_{2n}(\cos\vartheta) + (2n{+}1)\ \mathrm{P}_{2n}(\cos\vartheta)\,, \tag{126}$$

$$K_n^0(\cos\vartheta) = -\ \sin^2\!\vartheta\ \partial\mathrm{P}_{2n}(\cos\vartheta) + (2n{+}1)\ \cos\vartheta\ \mathrm{P}_{2n}(\cos\vartheta)\,, \tag{127}$$

and

$$\partial\mathrm{P}_n(u) = \partial_u\mathrm{P}_n(u)\,. \tag{128}$$

Using the formulae for the Legendre polynomials (see BE, 3.8, (19))

$$\sin^2\!\vartheta\ \partial\mathrm{P}_{2n}(\cos\vartheta) = (2n{+}1)\Big(\cos\vartheta\ \mathrm{P}_{2n}(\cos\vartheta) - \mathrm{P}_{2n+1}(\cos\vartheta)\Big)\,, \tag{129}$$

$$\sin^2\!\vartheta\ \partial\mathrm{P}_{2n+1}(\cos\vartheta) = (2n{+}1)\Big(\mathrm{P}_{2n}(\cos\vartheta) - \cos\vartheta\ \mathrm{P}_{2n+1}(\cos\vartheta)\Big)\,, \tag{130}$$

we get

$$K_n^1(\cos\vartheta) = \partial\mathrm{P}_{2n+1}(\cos\vartheta)\,, \tag{131}$$

$$K_n^0(\cos\vartheta) = (2n{+}1)\ \mathrm{P}_{2n+1}(\cos\vartheta)\,, \tag{132}$$

and thus

$$\boldsymbol{E}(\boldsymbol{r}) = \sum\nolimits_{n\ge 0} V_n\,\frac{a^{2n}}{r^{2n+2}}\Big(\partial\mathrm{P}_{2n+1}(\cos\vartheta)\ \sin\vartheta\ (\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi) + {} + \mathbf{k}\,(2n{+}1)\ \mathrm{P}_{2n+1}(\cos\vartheta)\Big)\,. \tag{133}$$

In order to find the relation between the expressions in ellipsoidal coordinates (120), (104) and the corresponding expressions in spherical coordinates (124), (133), we use the well known procedure (see for example *Heiskanen and Moritz (1967)*, Paragraph 2-9): we compare both expressions at the positive z-axis. According to (8) and (9) we have there $\cos\vartheta = 1$, $\cos\xi = 1$ and $r = a\sqrt{1-\varepsilon^2}\,\upsilon$. Using the equality $\mathrm{P}_n(1) = 1$ and (40) we get from (120) and (104) the formulae

$$V(r\,\mathbf{k}) = -\,\frac{\kappa M}{r}\left(\mathrm{r}_0^0\Big(\frac{r}{a\varepsilon}\Big) + \frac{2\,a^2\varepsilon^2}{15\,r^2}\,\mathrm{r}_2^0\Big(\frac{r}{a\varepsilon}\Big)\right), \tag{134}$$

$$\boldsymbol{E}(r\,\mathbf{k}) = -\,\frac{\kappa M}{r^2}\,\mathbf{k}\,\mathrm{r}_1^0\Big(\frac{r}{a\varepsilon}\Big)\,, \tag{135}$$

and using the formulae (94), (95) and (97) we obtain

$$V(r\,\mathbf{k}) = -\,\frac{\kappa M}{r}\sum\nolimits_{n\geq 0}\frac{3\,(-1)^n\,a^{2n}\varepsilon^{2n}}{(2n+1)(2n+3)\,r^{2n}}\,, \tag{136}$$

$$\boldsymbol{E}(r\,\mathbf{k}) = -\,\frac{\kappa M}{r^2}\,\mathbf{k}\sum\nolimits_{n\geq 0}\frac{3\,(-1)^n\,a^{2n}\varepsilon^{2n}}{(2n+3)\,r^{2n}}\,. \tag{137}$$

On the other hand, from (124) and (133) we have

$$V(r\,\mathbf{k}) = \sum\nolimits_{n\geq 0} V_n\,\frac{a^{2n}}{r^{2n+1}}\,, \tag{138}$$

$$\boldsymbol{E}(r\,\mathbf{k}) = \mathbf{k}\sum\nolimits_{n\geq 0}(2n+1)\,V_n\,\frac{a^{2n}}{r^{2n+2}}\,, \tag{139}$$

and comparing with (136) and (137) we obtain the values of the coefficients V_n:

$$V_n = -\,\kappa M\,\frac{3\,(-1)^n\,\varepsilon^{2n}}{(2n+1)(2n+3)}\,. \tag{140}$$

The formulae for the gravitational potential and intensity outside the body in spherical coordinates then read

$$V(\boldsymbol{r}) = -\,\frac{\kappa M}{r}\sum\nolimits_{n\geq 0}\frac{3\,(-1)^n\,\varepsilon^{2n}}{(2n+1)(2n+3)}\,\frac{a^{2n}}{r^{2n}}\;\mathrm{P}_{2n}(\cos\vartheta)\,, \tag{141}$$

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{r}) = -\,\frac{\kappa M}{r^2}\sum\nolimits_{n\geq 0}\frac{3\,(-1)^n\,\varepsilon^{2n}}{(2n+1)(2n+3)}\,\frac{a^{2n}}{r^{2n}}\,\cdot\\ \cdot\Big(\partial\mathrm{P}_{2n+1}(\cos\vartheta)\,\sin\vartheta\,(\mathbf{i}\cos\varphi+\mathbf{j}\sin\varphi)\,+\\ +\,\mathbf{k}\,(2n+1)\,\mathrm{P}_{2n+1}(\cos\vartheta)\Big)\,.\end{aligned} \tag{142}$$

6. Equilibrium state of a rotating homogeneous ellipsoidal body

Let us consider the case that the homogeneous ellipsoidal body is rotating about the z-axis with the angular velocity ω. Then any mass point fixed with respect to the body at the point $\boldsymbol{r}$ is subject (in the frame rotating with the body) to the centrifugal acceleration given by the formula

$$\boldsymbol{C}(\boldsymbol{r}) = -\,\omega^2\,\mathbf{k}\times(\mathbf{k}\times\boldsymbol{r})\,, \tag{143}$$

and thus the total acceleration of the mass point is

$$\boldsymbol{G}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{C}(\boldsymbol{r})\,. \tag{144}$$

In analogy with the connection between the gravitational potential $V(\boldsymbol{r})$ and the intensity $\boldsymbol{E}(\boldsymbol{r})$ (see (2)) we can introduce the (quasi)potential $W(\boldsymbol{r})$ such that

$$\boldsymbol{G}(\boldsymbol{r}) = -\,\nabla_{\boldsymbol{r}}W(\boldsymbol{r})\,, \tag{145}$$

and then we get

$$W(\boldsymbol{r}) = V(\boldsymbol{r}) + Z(\boldsymbol{r})\,, \tag{146}$$

where $Z(\boldsymbol{r})$ is the centrifugal potential:

$$Z(\boldsymbol{r}) = -\,\frac{1}{2}\,\omega^2\,(\mathbf{k}\times\boldsymbol{r})^2\,. \tag{147}$$

Using the formula (9) we obtain the expressions for the centrifugal acceleration $\boldsymbol{C}(\boldsymbol{r})$ and potential $Z(\boldsymbol{r})$ in ellipsoidal coordinates:

$$\boldsymbol{C}(\boldsymbol{r}) = \omega^2\,a\,\sqrt{(1-\varepsilon^2)\,v^2+\varepsilon^2}\,\sin\xi\,\left(\mathbf{i}\cos\psi + \mathbf{j}\sin\psi\right), \tag{148}$$

$$Z(\boldsymbol{r}) = -\,\frac{1}{2}\,\omega^2 a^2\left((1-\varepsilon^2)\,v^2+\varepsilon^2\right)\,\sin^2\!\xi\,. \tag{149}$$

The rotating ellipsoidal body is in equilibrium if the total acceleration at the surface of the body $\boldsymbol{G}(\boldsymbol{s})$ has no component tangential to the surface; in other words, if it is proportional to the normal vector of the surface $\boldsymbol{n}(\boldsymbol{s})$. Thus in equilibrium we have

$$\boldsymbol{G}(\boldsymbol{s}) = G(\boldsymbol{s})\,\boldsymbol{n}(\boldsymbol{s})\,, \tag{150}$$

where $G(\boldsymbol{s})$ is the normal component of the total acceleration. Using the formulae (17), (20), (21), (122), (144) and (148) we obtain the condition

$$-\frac{\kappa M}{a^2(1-\varepsilon^2)}\Big(\sqrt{1-\varepsilon^2}\,\mathrm{r}_1^1(\kappa(\varepsilon))\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+\mathbf{k}\,\mathrm{r}_1^0(\kappa(\varepsilon))\cos\xi\Big)+ \\ +\omega^2 a\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)= \\ =\frac{G(\boldsymbol{s})}{k(\boldsymbol{v})}\Big(\sqrt{1-\varepsilon^2}\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+\mathbf{k}\cos\xi\Big), \tag{151}$$

from which we easily get according to (40), (92) and (93) the formulae

$$G(\boldsymbol{s})=-\frac{\kappa M}{a^2(1-\varepsilon^2)}\,\mathrm{r}_1^0(\kappa(\varepsilon))\sqrt{1-\varepsilon^2\sin^2\xi}\,, \tag{152}$$

$$\omega^2=\frac{\kappa M}{a^3\sqrt{1-\varepsilon^2}}\,\frac{2\varepsilon^2}{5(1-\varepsilon^2)}\,\mathrm{r}_2^0(\kappa(\varepsilon))\,. \tag{153}$$

The total acceleration at the surface S is then given by the formula

$$\boldsymbol{G}(\boldsymbol{s})=-\frac{\kappa M}{a^2(1-\varepsilon^2)}\,\mathrm{r}_1^0(\kappa(\varepsilon))\Big(\sqrt{1-\varepsilon^2}\sin\xi\,(\mathbf{i}\cos\psi+\mathbf{j}\sin\psi)+ \\ +\mathbf{k}\cos\xi\Big)\,; \tag{154}$$

for the centrifugal potential at the surface we get from (149) and (153)

$$Z(\boldsymbol{s})=-\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\,\frac{2\varepsilon^2}{15(1-\varepsilon^2)}\,\mathrm{r}_2^0(\kappa(\varepsilon))\Big(1-\mathrm{P}_2(\cos\xi)\Big), \tag{155}$$

and for the total potential at the surface we obtain according to (123) and (146)

$$W(\boldsymbol{s})=-\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\left(\mathrm{r}_0^0(\kappa(\varepsilon))+\frac{2\varepsilon^2}{15(1-\varepsilon^2)}\,\mathrm{r}_2^0(\kappa(\varepsilon))\right). \tag{156}$$

We see that the total potential at the surface S is constant as expected in equilibrium.

Now we compare the field of the homogeneous ellipsoidal body in equilibrium with the so-called normal gravity field which is defined as follows (see *Heiskanen and Moritz (1967)*, Paragraph 2-7):

– the normal gravitational potential $V_{\rm N}(\boldsymbol{r})$ is defined in the exterior of the ellipsoidal body $D_{\rm ext}$ and on its surface S,
– the potential $V_{\rm N}(\boldsymbol{r})$ is harmonic in $D_{\rm ext}$ and the limit of the quantity $r\,V_{\rm N}(\boldsymbol{r})$ for $r \to \infty$ is equal to $-\,\kappa M$,
– the normal gravity potential $W_{\rm N}(\boldsymbol{r}) = V_{\rm N}(\boldsymbol{r}) + Z(\boldsymbol{r})$ for the given angular velocity ω is constant at the surface S.

Comparing the formulae (123) and (156) and taking into account the formula (153) we immediately obtain

$$V_{\rm N}(\boldsymbol{s}) = -\,\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\,{\rm r}_0^0(\kappa(\varepsilon)) - \frac{1}{3}\,\omega^2 a^2\,{\rm P}_2(\cos\xi)\,, \tag{157}$$

$$W_{\rm N}(\boldsymbol{s}) = -\,\frac{\kappa M}{a\sqrt{1-\varepsilon^2}}\,{\rm r}_0^0(\kappa(\varepsilon)) - \frac{1}{3}\,\omega^2 a^2\,, \tag{158}$$

and thus according to (42), (85) and (87) we get for $\boldsymbol{r} \in D_{\rm ext}$

$$V_{\rm N}(\boldsymbol{r}) = -\,\frac{\kappa M}{a\sqrt{1-\varepsilon^2}\,\upsilon}\,{\rm r}_0^0(\kappa(\varepsilon)\,\upsilon) - \frac{1}{3}\,\omega^2 a^2\,\frac{1}{\upsilon^3}\,\frac{{\rm r}_2^0(\kappa(\varepsilon)\,\upsilon)}{{\rm r}_2^0(\kappa(\varepsilon))}\,{\rm P}_2(\cos\xi)\,. \tag{159}$$

We see that for the angular velocity ω given by (153) the expressions for the potentials $V(\boldsymbol{r})$ (120) and $V_{\rm N}(\boldsymbol{r})$ (159) coincide.

The main problem with the normal gravity field is that it assumes that the body has the shape of the rotational ellipsoid and that it is (at the surface) in equilibrium, but it does not bother about the source of the gravitational field – the density distribution in the interior of the body. There arises a question as to what can be the possible density distribution within the body which generates the adopted external gravitational field; we shall touch this point in the next Section.

7. Gravitational field of a layered ellipsoidal body

We shall define the layered ellipsoidal body as a set of homogeneous bodies with the shape of the rotational ellipsoid such that all bodies have the common centre and rotational axis. Let the number of the bodies be $I+1$, where $I \geq 0$; then for any i such that $0 \leq i \leq I$, let S_i be the surface of the i-th body and D_i its interior. We define the set of bodies in such a way that the (i+1)-th body is fully contained within the interior of the i-th body (thus $D_{i+1} \subset D_i$); then the i-th layer is defined as the set of points $\boldsymbol{r}$ lying in the interior of the i-th body but in the exterior of the (i+1)-th body (thus $\boldsymbol{r} \in D_i$, $\boldsymbol{r} \notin D_{i+1} \cup S_{i+1}$). Further, let the i-th body have the equatorial radius a_i, excentricity ε_i and density σ_i; according to the previous sentence, the dimension parameters have to satisfy for each i ($0 \leq i \leq I-1$) the inequalities

$$a_{i+1} < a_i\,, \quad a_{i+1}\sqrt{1-\varepsilon_{i+1}^2} < a_i\sqrt{1-\varepsilon_i^2}\,. \tag{160}$$

Our definition of the layered ellipsoidal body means that the actual density of the body $\rho(\boldsymbol{r})$ at the point $\boldsymbol{r}$ lying in the i-th layer is equal to

$$\rho(\boldsymbol{r}) = \sum\nolimits_{0\leq k\leq i} \sigma_k\,. \tag{161}$$

Finally, we shall require that the density $\rho(\boldsymbol{r})$ increases if we cross any surface S_i towards the centre; this means that for each i ($0 \leq i \leq I$) we have $\sigma_i > 0$.

As we do not restrict the values of the parameters a_i and ε_i besides the conditions (160), it is not useful to use the ellipsoidal coordinates for the expression of the gravitational potential and intensity of the particular bodies. The reason is that the ellipsoidal surface with equatorial radius a_i and excentricity ε_i is in the ellipsoidal coordinates a surface given by the constant value of the radial coordinate υ only if $a_i\varepsilon_i = a\varepsilon$. Therefore we transform our expressions from Section 5 into the spherical coordinates using the formulae (14) and we replace the parameters a and ε by the parameters b and e according to the definition (12). Then we obtain the replacements

$$a = \sqrt{b^2+e^2}\,, \quad a\sqrt{1-\varepsilon^2} = b\,, \quad a\sqrt{1-\varepsilon^2}\,\upsilon = s(\boldsymbol{r},e)\,, \tag{162}$$

(where $s(\boldsymbol{r},e)$ is given by (13)) and comparing the formulae (8) and (9) we obtain the replacements

$$a\sqrt{(1-\varepsilon^2)\,\upsilon^2+\varepsilon^2}\,\sin\xi = r\sin\vartheta\,, \tag{163}$$

$$a\sqrt{1-\varepsilon^2}\,\upsilon\cos\xi = r\cos\vartheta\,. \tag{164}$$

For the intensity we get after straightforward calculation (using (40)) from (103) the formula valid for $s(\boldsymbol{r}, e) < b$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{\kappa M}{b^2}\left(\mathrm{r}_1^1\Big(\frac{b}{e}\Big)\,\frac{b\,r\,\sin\vartheta}{b^2+e^2}\,(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi) + \right.$$
$$\left. + \mathbf{k}\,\mathrm{r}_1^0\Big(\frac{b}{e}\Big)\,\frac{r\,\cos\vartheta}{b}\right), \tag{165}$$

and from (104) the formula valid for $s(\boldsymbol{r}, e) \geq b$

$$\boldsymbol{E}(\boldsymbol{r}) = -\frac{\kappa M}{s(\boldsymbol{r},e)^2}\left(\mathrm{r}_1^1\Big(\frac{s(\boldsymbol{r},e)}{e}\Big)\,\frac{s(\boldsymbol{r},e)\,r\,\sin\vartheta}{s(\boldsymbol{r},e)^2+e^2}\,(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi) + \right.$$
$$\left. + \mathbf{k}\,\mathrm{r}_1^0\Big(\frac{s(\boldsymbol{r},e)}{e}\Big)\,\frac{r\,\cos\vartheta}{s(\boldsymbol{r},e)}\right). \tag{166}$$

For the potential we get according to (64) from (105) and (107) the formula valid for $s(\boldsymbol{r}, e) < b$

$$V(\boldsymbol{r}) = -\frac{\kappa M}{b}\,\frac{1}{2}\left(3\,\mathrm{r}_0^0\Big(\frac{b}{e}\Big) - \mathrm{r}_1^1\Big(\frac{b}{e}\Big)\,\frac{r^2\sin^2\vartheta}{b^2+e^2} - \mathrm{r}_1^0\Big(\frac{b}{e}\Big)\,\frac{r^2\cos^2\vartheta}{b^2}\right), \tag{167}$$

and from (106) and (108) the formula valid for $s(\boldsymbol{r}, e) \geq b$

$$V(\boldsymbol{r}) = -\frac{\kappa M}{s(\boldsymbol{r},e)}\,\frac{1}{2}\left(3\,\mathrm{r}_0^0\Big(\frac{s(\boldsymbol{r},e)}{e}\Big) - \mathrm{r}_1^1\Big(\frac{s(\boldsymbol{r},e)}{e}\Big)\,\frac{r^2\sin^2\vartheta}{s(\boldsymbol{r},e)^2+e^2} - \right.$$
$$\left. - \mathrm{r}_1^0\Big(\frac{s(\boldsymbol{r},e)}{e}\Big)\,\frac{r^2\cos^2\vartheta}{s(\boldsymbol{r},e)^2}\right). \tag{168}$$

The mass M is given according to (102) by

$$M = \frac{4\pi}{3}\,\rho\,(b^2+e^2)\,b\,. \tag{169}$$

Comparing the formulae for the intensity and potential in the interior and exterior of the body we see that we can write them in a unified form: after introducing the function $s(\boldsymbol{r}, b, e)$

$$s(\boldsymbol{r},e) < b: \quad s(\boldsymbol{r},b,e) = b\,,$$
$$s(\boldsymbol{r},e) \geq b: \quad s(\boldsymbol{r},b,e) = s(\boldsymbol{r},e)\,, \tag{170}$$

we write the intensity as $\boldsymbol{E}(\boldsymbol{r}, M, b, e)$ and we obtain from (165) and (166) the expression

$$\boldsymbol{E}(\boldsymbol{r}, M, b, e) = -\frac{\kappa M}{s(\boldsymbol{r}, b, e)^2} \cdot \cdot \left(\mathrm{r}_1^1\Big(\frac{s(\boldsymbol{r}, b, e)}{e}\Big) \frac{s(\boldsymbol{r}, b, e)\; r \sin\vartheta}{s(\boldsymbol{r}, b, e)^2 + e^2} \left(\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi\right) + + \mathbf{k}\, \mathrm{r}_1^0\Big(\frac{s(\boldsymbol{r}, b, e)}{e}\Big) \frac{r \cos\vartheta}{s(\boldsymbol{r}, b, e)} \right); \quad (171)$$

similarly, we write the potential as $V(\boldsymbol{r}, M, b, e)$ and we obtain from (167) and (168) the expression

$$V(\boldsymbol{r}, M, b, e) = -\frac{\kappa M}{s(\boldsymbol{r}, b, e)} \cdot \cdot \frac{1}{2} \left(3\, \mathrm{r}_0^0\Big(\frac{s(\boldsymbol{r}, b, e)}{e}\Big) - \mathrm{r}_1^1\Big(\frac{s(\boldsymbol{r}, b, e)}{e}\Big) \frac{r^2 \sin^2\vartheta}{s(\boldsymbol{r}, b, e)^2 + e^2} - - \mathrm{r}_1^0\Big(\frac{s(\boldsymbol{r}, b, e)}{e}\Big) \frac{r^2 \cos^2\vartheta}{s(\boldsymbol{r}, b, e)^2} \right), \quad (172)$$

where the mass M is given by (169).

Now we can express the gravitational field of the layered ellipsoidal body in a simple form: for the intensity we have

$$\boldsymbol{E}(\boldsymbol{r}) = \sum\nolimits_{0 \le i \le I} \boldsymbol{E}(\boldsymbol{r}, M_i, b_i, e_i)\,, \quad (173)$$

and for the potential

$$V(\boldsymbol{r}) = \sum\nolimits_{0 \le i \le I} V(\boldsymbol{r}, M_i, b_i, e_i)\,; \quad (174)$$

the total mass of the layered ellipsoidal body is given by

$$M = \sum\nolimits_{0 \le i \le I} M_i\,, \quad (175)$$

where according to (169)

$$M_i = \frac{4\pi}{3}\, \sigma_i\, (b_i^2 + e_i^2)\, b_i\,. \quad (176)$$

After obtaining the expression for the gravitational field of the layered ellipsoidal body we can turn to the problem of the equilibrium state of such a body; we assume that the body rotates as a whole with the angular velocity ω. The total potential $W(\boldsymbol{r})$ is given by (146), where the gravitational potential $V(\boldsymbol{r})$ is given by (174) and the centrifugal potential $Z(\boldsymbol{r})$ is according to (147) equal to

$$Z(\boldsymbol{r}) = -\frac{1}{2}\,\omega^2\, r^2 \sin^2\vartheta\,. \tag{177}$$

On the contrary to the homogeneous body we have to satisfy the equilibrium conditions at each boundary surface S_i. In order to formulate the condition for some particular i ($0 \le i \le I$), we introduce the auxiliary ellipsoidal coordinate system with coordinates $s_{\rm a}$, $\xi_{\rm a}$, $\psi_{\rm a}$ in which the radius-vector $\boldsymbol{r}$ can be expressed according to (9) and (12) as

$$\boldsymbol{r} = \sqrt{s_{\rm a}^2 + e_i^2}\,\sin\xi_{\rm a}\,(\mathbf{i}\cos\psi_{\rm a} + \mathbf{j}\sin\psi_{\rm a}) + \mathbf{k}\, s_{\rm a}\,\cos\xi_{\rm a}\,; \tag{178}$$

note that the coordinate system is different for different values of i (we avoided the index i in the coordinates, as such notation could be misinterpreted as a particular constant value of the corresponding coordinate). The parametric expression of the radius-vector $\boldsymbol{s}_i(\xi_{\rm a},\psi_{\rm a})$ of a point at the boundary surface S_i can be obtained by replacing $s_{\rm a}$ by b_i:

$$\boldsymbol{s}_i(\xi_{\rm a},\psi_{\rm a}) = \sqrt{b_i^2 + e_i^2}\,\sin\xi_{\rm a}\,(\mathbf{i}\cos\psi_{\rm a} + \mathbf{j}\sin\psi_{\rm a}) + \mathbf{k}\, b_i\,\cos\xi_{\rm a}\,. \tag{179}$$

In order to calculate the potentials $V(\boldsymbol{r})$ and $Z(\boldsymbol{r})$ according to (172), (174) and (177), we still have to know the values of quantities $r^2\sin^2\vartheta$ and $r^2\cos^2\vartheta$ at the surface S_i: we evidently have

$$r^2\sin^2\vartheta = (b_i^2 + e_i^2)\,\sin^2\xi_{\rm a}\,, \tag{180}$$

$$r^2\cos^2\vartheta = b_i^2\cos^2\xi_{\rm a}\,. \tag{181}$$

Inserting the expression (179) for $\boldsymbol{r}$ in the formulae (174) and (177) and using (180) and (181) we obtain from (146) the equilibrium condition ($0 \le i \le I$)

$$\sum\nolimits_{0\le k\le I} V(\boldsymbol{s}_i(\xi_{\rm a},\psi_{\rm a}), M_k, b_k, e_k) + Z(\boldsymbol{s}_i(\xi_{\rm a},\psi_{\rm a})) = W_i\,, \tag{182}$$

where W_i is the unknown constant. As the values of parameters e_k for $k \ne i$ are in general different from e_i, the equilibrium condition is very complicated and it is unlikely that the l.h.s. of (182) can be expressed as a series of some base functions of ξ_a (for example the Legendre polynomials $\mathrm{P}_n(\cos\xi_\mathrm{a})$) whose coefficients have a manageable form.

The solution of this problem may be found in the following way: both sides of the condition (182) are functions defined at the surface S_i; instead of these we shall compare the functions harmonic in the exterior of the i-th body whose values at the surface S_i are the l.h.s. and the r.h.s. of (182) (more exactly, we shall compare the values of these functions at the positive z-axis). The construction of these harmonic functions is straightforward.

The sum on the l.h.s. of (182) can be divided into two sums, the first one for $0 \le k \le i$–1 and the second one for $i \le k \le I$. The second sum represents the contribution of the i-th body and of bodies which are contained in the interior of the i-th body; therefore the contribution of each of these bodies has the form of the external potential (note that for $i \le k \le I$ we have $s(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}), e_k) \ge b_k$ and according to (170) the potential (172) has the form (168)).

On the contrary, the first sum represents the contribution of bodies whose interior contains the i-th body; therefore the contribution of each of these bodies has the form of the internal potential (as for $0 \le k \le i-1$ we have $s(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}), e_k) < b_k$ and according to (170) the potential (172) has the form (167)). The contribution of each of these bodies (and also of the centrifugal potential (177) and of the constant W_i) is a linear combination of 1 and quantities $r^2\sin^2\vartheta$ and $r^2\cos^2\vartheta$ at the surface S_i (the latter are given by (180) and (181)). Therefore we introduce the functions $h_0(\boldsymbol{r}, b_i, e_i)$, $h_1(\boldsymbol{r}, b_i, e_i)$ and $h_2(\boldsymbol{r}, b_i, e_i)$ which are harmonic for $s(\boldsymbol{r}, e_i) \ge b_i$ and which satisfy at the surface S_i (thus for $s(\boldsymbol{r}, e_i) = b_i$) the conditions

$$h_0(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}), b_i, e_i) = 1\,, \tag{183}$$

$$h_1(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}), b_i, e_i) = (b_i^2 + e_i^2)\sin^2\xi_\mathrm{a} = \frac{2}{3}\,(b_i^2 + e_i^2)\Big(1 - \mathrm{P}_2(\cos\xi_\mathrm{a})\Big)\,, \tag{184}$$

$$h_2(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}), b_i, e_i) = b_i^2\cos^2\xi_\mathrm{a} = \frac{1}{3}\,b_i^2\Big(1 + 2\,\mathrm{P}_2(\cos\xi_\mathrm{a})\Big)\,, \tag{185}$$

where we used the formula (109). Taking into account the formulae (42), (44), (85), (87), (162) and (178) we can directly write

$$h_0(\boldsymbol{r}, b_i, e_i) = \frac{b_i}{s(\boldsymbol{r}, e_i)}\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{s(\boldsymbol{r}, e_i)}{e_i}\Big)\,, \tag{186}$$

$$h_1(\boldsymbol{r}, b_i, e_i) = \frac{2}{3}\,\frac{(b_i^2 + e_i^2)\, b_i}{s(\boldsymbol{r}, e_i)} \left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\, \mathrm{r}_0^0\Big(\frac{s(\boldsymbol{r}, e_i)}{e_i}\Big) - {} - \frac{b_i^2}{s(\boldsymbol{r}, e_i)^2}\, \overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\, \mathrm{r}_2^0\Big(\frac{s(\boldsymbol{r}, e_i)}{e_i}\Big)\, \mathrm{P}_2(\cos \xi_{\mathrm{a}}) \right), \quad (187)$$

$$h_2(\boldsymbol{r}, b_i, e_i) = \frac{1}{3}\,\frac{b_i^3}{s(\boldsymbol{r}, e_i)} \left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\, \mathrm{r}_0^0\Big(\frac{s(\boldsymbol{r}, e_i)}{e_i}\Big) + {} + 2\, \frac{b_i^2}{s(\boldsymbol{r}, e_i)^2}\, \overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\, \mathrm{r}_2^0\Big(\frac{s(\boldsymbol{r}, e_i)}{e_i}\Big)\, \mathrm{P}_2(\cos \xi_{\mathrm{a}}) \right), \quad (188)$$

where we introduced for brevity

$$\overline{\mathrm{r}_0^0}(u) = \frac{1}{\mathrm{r}_0^0(u)}\,, \quad \overline{\mathrm{r}_2^0}(u) = \frac{1}{\mathrm{r}_2^0(u)}\,. \quad (189)$$

Now we replace on the r.h.s. of the formula (167) the terms 1, $r^2 \sin^2\vartheta$, $r^2 \cos^2\vartheta$ by the functions $h_0(\boldsymbol{r}, b_i, e_i)$, $h_1(\boldsymbol{r}, b_i, e_i)$, $h_2(\boldsymbol{r}, b_i, e_i)$, respectively; the function $V(\boldsymbol{r}, M, b, e, b_i, e_i)$ obtained in this way is harmonic for $s(\boldsymbol{r}, e_i) \geq b_i$:

$$V(\boldsymbol{r}, M, b, e, b_i, e_i) = -\frac{\kappa M}{b}\,\frac{1}{2} \left(3\, \mathrm{r}_0^0\Big(\frac{b}{e}\Big)\, h_0(\boldsymbol{r}, b_i, e_i) - {} - \mathrm{r}_1^1\Big(\frac{b}{e}\Big)\, \frac{h_1(\boldsymbol{r}, b_i, e_i)}{b^2 + e^2} - \mathrm{r}_1^0\Big(\frac{b}{e}\Big)\, \frac{h_2(\boldsymbol{r}, b_i, e_i)}{b^2} \right). \quad (190)$$

According to (170), (172), (180), (181), (183), (184) and (185), this function satisfies at the surface S_i for any k such that $0 \leq k \leq i-1$ the equality

$$V(\boldsymbol{s}_i(\xi_{\mathrm{a}}, \psi_{\mathrm{a}}), M_k, b_k, e_k, b_i, e_i) = V(\boldsymbol{s}_i(\xi_{\mathrm{a}}, \psi_{\mathrm{a}}), M_k, b_k, e_k)\,. \quad (191)$$

Similarly, using the formulae (177), (180), (183) and (184) we define the following functions harmonic for $s(\boldsymbol{r}, e_i) \geq b_i$:

$$Z(\boldsymbol{r}, b_i, e_i) = -\frac{1}{2}\, \omega^2\, h_1(\boldsymbol{r}, b_i, e_i)\,, \quad (192)$$

$$W_i(\boldsymbol{r}, b_i, e_i) = W_i\; h_0(\boldsymbol{r}, b_i, e_i)\,, \quad (193)$$

which evidently satisfy at the surface S_i the equalities

$$Z(\boldsymbol{s}_i(\xi_{\mathrm{a}}, \psi_{\mathrm{a}}), b_i, e_i) = Z(\boldsymbol{s}_i(\xi_{\mathrm{a}}, \psi_{\mathrm{a}}))\,, \quad (194)$$

$$W_i(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}),b_i,e_i) = W_i\,. \tag{195}$$

We are now ready to rewrite the condition (182) using the formulae (191), (194) and (195) ($0 \le i \le I$):

$$\begin{aligned}\textstyle\sum_{0\le k\le i-1} V(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}),M_k,b_k,e_k,b_i,e_i) + \\ + \textstyle\sum_{i\le k\le I} V(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}),M_k,b_k,e_k) + \\ + Z(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}),b_i,e_i) = W_i(\boldsymbol{s}_i(\xi_\mathrm{a},\psi_\mathrm{a}),b_i,e_i)\,;\end{aligned} \tag{196}$$

as each term is a harmonic function for $s(\boldsymbol{r},e_i) \ge b_i$, we get the equation (valid for $s(\boldsymbol{r},e_i) \ge b_i$)

$$\begin{aligned}\textstyle\sum_{0\le k\le i-1} V(\boldsymbol{r},M_k,b_k,e_k,b_i,e_i) + \sum_{i\le k\le I} V(\boldsymbol{r},M_k,b_k,e_k) + \\ + Z(\boldsymbol{r},b_i,e_i) = W_i(\boldsymbol{r},b_i,e_i)\,.\end{aligned} \tag{197}$$

In particular, we obtain at the positive z-axis the equation (valid for $r \ge b_i$, $0 \le i \le I$)

$$\begin{aligned}\textstyle\sum_{0\le k\le i-1} V(r\,\mathbf{k},M_k,b_k,e_k,b_i,e_i) + \sum_{i\le k\le I} V(r\,\mathbf{k},M_k,b_k,e_k) + \\ + Z(r\,\mathbf{k},b_i,e_i) = W_i(r\,\mathbf{k},b_i,e_i)\,.\end{aligned} \tag{198}$$

According to (13) we have $s(r\,\mathbf{k},e) = r$; using (92) we obtain from (170) and (172) for $r \ge b_k$

$$V(r\,\mathbf{k},M_k,b_k,e_k) = -\frac{\kappa M_k}{r}\,\mathrm{r}_1^1\Big(\frac{r}{e_k}\Big)\,. \tag{199}$$

In a similar way we obtain from (186), (187) and (188) for $r \ge b_i$

$$h_0(r\,\mathbf{k},b_i,e_i) = \frac{b_i}{r}\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big), \tag{200}$$

$$h_1(r\,\mathbf{k},b_i,e_i) = \frac{2}{3}\,\frac{(b_i^2+e_i^2)\,b_i}{r}\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right), \tag{201}$$

$$h_2(r\,\mathbf{k},b_i,e_i) = \frac{1}{3}\,\frac{b_i^3}{r}\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) + 2\,\frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right), \tag{202}$$

and inserting in (190), (192) and (193) we get

$$V(r\,\mathbf{k}, M_k, b_k, e_k, b_i, e_i) = -\,\frac{\kappa\,M_k}{r}\,\frac{1}{2}\,\frac{b_i}{b_k}\left(3\,\mathrm{r}_0^0\Big(\frac{b_k}{e_k}\Big)\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \right.$$
$$-\,\frac{2}{3}\,\frac{b_i^2+e_i^2}{b_k^2+e_k^2}\,\mathrm{r}_1^1\Big(\frac{b_k}{e_k}\Big)\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right) -$$
$$\left. -\,\frac{1}{3}\,\frac{b_i^2}{b_k^2}\,\mathrm{r}_1^0\Big(\frac{b_k}{e_k}\Big)\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) + 2\,\frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right)\right), \qquad (203)$$

$$Z(r\,\mathbf{k}, b_i, e_i) = -\,\frac{1}{3}\,\omega^2\,\frac{(b_i^2+e_i^2)\,b_i}{r}\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \right.$$
$$\left. -\,\frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right), \qquad (204)$$

$$W_i(r\,\mathbf{k}, b_i, e_i) = W_i\,\frac{b_i}{r}\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big)\,. \qquad (205)$$

The condition (198) acquires the form ($0 \le i \le I$)

$$\sum\nolimits_{0\le k\le i-1} \kappa\,M_k\,\frac{1}{2}\,\frac{b_i}{b_k}\left(3\,\mathrm{r}_0^0\Big(\frac{b_k}{e_k}\Big)\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \right.$$
$$-\,\frac{2}{3}\,\frac{b_i^2+e_i^2}{b_k^2+e_k^2}\,\mathrm{r}_1^1\Big(\frac{b_k}{e_k}\Big)\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right) -$$
$$\left. -\,\frac{1}{3}\,\frac{b_i^2}{b_k^2}\,\mathrm{r}_1^0\Big(\frac{b_k}{e_k}\Big)\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) + 2\,\frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right)\right) +$$
$$+ \sum\nolimits_{i\le k\le I} \kappa\,M_k\,\mathrm{r}_1^1\Big(\frac{r}{e_k}\Big) +$$
$$+\,\frac{1}{3}\,\omega^2\,(b_i^2+e_i^2)\,b_i\left(\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) - \frac{b_i^2}{r^2}\,\overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_2^0\Big(\frac{r}{e_i}\Big)\right) +$$
$$+\,W_i\,b_i\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,\mathrm{r}_0^0\Big(\frac{r}{e_i}\Big) = 0\,, \qquad (206)$$

and inserting the series (94), (96) and (97) we get the set of conditions (for $n \ge 0$, $0 \le i \le I$)

$$(2\,n+3)\Big(A_i - B_i - C_i + O_i + W_i\,b_i\Big)\,\overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big)\,e_i^{2n} +$$

$$+ 15\, n \Big(- B_i + 2\, C_i + O_i\Big)\, \overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\, e_i^{2n-2}\, b_i^2 +$$

$$+ \sum\nolimits_{i \le k \le I} 3\, \kappa\, M_k\, e_k^{2n} = 0\,, \tag{207}$$

where we denoted

$$A_i = \sum\nolimits_{0 \le k \le i-1} \kappa\, M_k\, \frac{3}{2}\, \frac{b_i}{b_k}\, \mathrm{r}_0^0\Big(\frac{b_k}{e_k}\Big)\,, \tag{208}$$

$$B_i = \sum\nolimits_{0 \le k \le i-1} \kappa\, M_k\, \frac{1}{3}\, \frac{(b_i^2 + e_i^2)\, b_i}{(b_k^2 + e_k^2)\, b_k}\, \mathrm{r}_1^1\Big(\frac{b_k}{e_k}\Big)\,, \tag{209}$$

$$C_i = \sum\nolimits_{0 \le k \le i-1} \kappa\, M_k\, \frac{1}{6}\, \frac{b_i^3}{b_k^3}\, \mathrm{r}_1^0\Big(\frac{b_k}{e_k}\Big)\,, \tag{210}$$

$$O_i = \frac{1}{3}\, \omega^2\, (b_i^2 + e_i^2)\, b_i\,. \tag{211}$$

Inserting $n = 0$ in (207) we obtain the equation

$$\Big(A_i - B_i - C_i + O_i + W_i\, b_i\Big)\, \overline{\mathrm{r}_0^0}\Big(\frac{b_i}{e_i}\Big) + \sum\nolimits_{i \le k \le I} \kappa\, M_k = 0\,; \tag{212}$$

if we denote

$$N_i = \sum\nolimits_{i \le k \le I} \kappa\, M_k\,, \tag{213}$$

and use (189), we get the expression for the constant W_i $(0 \le i \le I)$

$$W_i = -\frac{1}{b_i}\left(A_i - B_i - C_i + O_i + N_i\, \mathrm{r}_0^0\Big(\frac{b_i}{e_i}\Big)\right) \tag{214}$$

and the set of equations (207) acquires the form $(n \ge 1,\ 0 \le i \le I)$

$$-\,(2\, n + 3)\, N_i\, e_i^{2n} + 15\, n \Big(- B_i + 2\, C_i + O_i\Big)\, \overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\, e_i^{2n-2}\, b_i^2 +$$

$$+ \sum\nolimits_{i \le k \le I} 3\, \kappa\, M_k\, e_k^{2n} = 0\,. \tag{215}$$

Now we put $n = 1$ and obtain the equation

$$-\,5\, N_i\, e_i^2 + 15 \Big(- B_i + 2\, C_i + O_i\Big)\, \overline{\mathrm{r}_2^0}\Big(\frac{b_i}{e_i}\Big)\, b_i^2 +$$

$$+ \sum\nolimits_{i \le k \le I} 3\, \kappa\, M_k\, e_k^2 = 0\,; \tag{216}$$

if we denote

$$E_i = \sum\nolimits_{i\le k\le I} \kappa\, M_k\, e_k^2\,, \tag{217}$$

and use (189), we get the expression for the quantity O_i given by (211) $(0 \le i \le I)$

$$O_i = \frac{5\, N_i\, e_i^2 - 3\, E_i}{15\, b_i^2}\, \mathrm{r}_2^0\Big(\frac{b_i}{e_i}\Big) + B_i - 2\, C_i \tag{218}$$

and the set of equations (215) acquires the form $(n \ge 2,\ 0 \le i \le I)$

$$(n-1)\, N_i\, e_i^{2n} - n\, E_i\, e_i^{2n-2} + \sum\nolimits_{i\le k\le I} \kappa\, M_k\, e_k^{2n} = 0\,. \tag{219}$$

We finally put $n = 2$ and obtain the equation

$$N_i\, e_i^4 - 2\, E_i\, e_i^2 + \sum\nolimits_{i\le k\le I} \kappa\, M_k\, e_k^4 = 0\,, \tag{220}$$

which can be according to (213) and (217) written in the form

$$\sum\nolimits_{i\le k\le I} \kappa\, M_k \left(e_k^2 - e_i^2\right)^2 = 0\,, \tag{221}$$

and this has the consequence $(0 \le i \le I)$

$$i{+}1 \le k \le I: \quad e_k = e_i\,, \tag{222}$$

as for all i $(0 \le i \le I)$ we have $M_i > 0$ (this follows from (176) and our assumption $\sigma_i > 0$). If (222) holds true, we obtain from (213) and (217)

$$E_i = N_i\, e_i^2\,, \tag{223}$$

and the condition (219) is satisfied for any $n \ge 2$. Thus, instead of the whole set of equations (215) $(n \ge 1,\ 0 \le i \le I)$ there remains the single equation (218) $(0 \le i \le I)$, which acquires the form

$$O_i = \frac{2\, e_i^2}{15\, b_i^2}\, \mathrm{r}_2^0\Big(\frac{b_i}{e_i}\Big)\, N_i + B_i - 2\, C_i\,. \tag{224}$$

Further, if the equalities (222) hold true for $i = 0$, they hold true for all i $(0 \le i \le I)$; thus all parameters e_i are mutually equal and we can put

$$0 \le i \le I: \quad e_i = e\,. \tag{225}$$

In other words, if there is an equilibrium for the rotating layered ellipsoidal body, then all boundaries S_i have to be rotational ellipsoids with the common focal circle. This is a rather unexpected result.

Of course, we still have to satisfy condition (224) for all $0 \leq i \leq I$; we rewrite it using the formulae (209), (210), (211) and (213) as $(0 \leq i \leq I)$

$$\omega^2 \, (b_i^2 + e^2) \, b_i = \frac{2\,e^2}{5\,b_i^2} \, \mathrm{r}_2^0\Big(\frac{b_i}{e}\Big) \sum\nolimits_{i \leq k \leq I} \kappa \, M_k + \\ + \sum\nolimits_{0 \leq k \leq i-1} \kappa \, M_k \, \frac{b_i}{b_k} \left(\frac{b_i^2 + e^2}{b_k^2 + e^2} \, \mathrm{r}_1^1\Big(\frac{b_k}{e}\Big) - \frac{b_i^2}{b_k^2} \, \mathrm{r}_1^0\Big(\frac{b_k}{e}\Big) \right). \quad (226)$$

For $i = 0$ we get using (175) the formula

$$\omega^2 = \frac{\kappa \, M}{(b_0^2 + e^2) \, b_0} \, \frac{2\,e^2}{5\,b_0^2} \, \mathrm{r}_2^0\Big(\frac{b_0}{e}\Big), \quad (227)$$

which is according to (162) for $b_0 = b$ identical with (153). For $1 \leq i \leq I$ we first derive from (92) and (93) the formula

$$\mathrm{r}_1^1(u) = \mathrm{r}_1^0(u) + \frac{2}{5\,u^2} \, \mathrm{r}_2^0(u), \quad (228)$$

and inserting in (226) we obtain

$$\omega^2 = \frac{1}{b_i^2 + e^2} \, \frac{2\,e^2}{5\,b_i^3} \, \mathrm{r}_2^0\Big(\frac{b_i}{e}\Big) \sum\nolimits_{i \leq k \leq I} \kappa \, M_k + \\ + \sum\nolimits_{0 \leq k \leq i-1} \kappa \, M_k \, \frac{1}{b_k^2 + e^2} \, \frac{2\,e^2}{5\,b_k^3} \, \mathrm{r}_2^0\Big(\frac{b_k}{e}\Big) + \\ + \frac{1}{b_i^2 + e^2} \sum\nolimits_{0 \leq k \leq i-1} \kappa \, M_k \, \frac{b_k^2 - b_i^2}{b_k^2 + e^2} \, \frac{e^2}{b_k^3} \, \mathrm{r}_1^0\Big(\frac{b_k}{e}\Big). \quad (229)$$

Now we use the fact that the function $\mathrm{q}_2^0(u)$ is according to (76) decreasing function of u (see also P99, Section 3); thus according to (87) the same is true for the function $\mathrm{r}_2^0(u)/u^3$. Further, as functions $\mathrm{q}_1^0(u)$ and $\mathrm{q}_2^0(u)$ are always positive, according to (86) and (87) the same is true for $u > 0$ for the functions $\mathrm{r}_1^0(u)$ and $\mathrm{r}_2^0(u)$. From (160) and (162) we get for each i $(0 \leq i \leq I-1)$ the inequality $b_{i+1} < b_i$; the inequality $M_i > 0$ for all i

$(0 \le i \le I)$ was mentioned above. Therefore we can write for $1 \le i \le I$ the inequality

$$\frac{1}{b_i^2+e^2}\,\frac{2\,e^2}{5\,b_i^3}\,\mathrm{r}_2^0\Big(\frac{b_i}{e}\Big)\sum\nolimits_{i\le k\le I}\kappa M_k > \\ > \frac{1}{b_0^2+e^2}\,\frac{2\,e^2}{5\,b_0^3}\,\mathrm{r}_2^0\Big(\frac{b_0}{e}\Big)\sum\nolimits_{i\le k\le I}\kappa M_k \tag{230}$$

(as $b_i < b_0$) and the inequality

$$\sum\nolimits_{0\le k\le i-1}\kappa M_k\,\frac{1}{b_k^2+e^2}\,\frac{2\,e^2}{5\,b_k^3}\,\mathrm{r}_2^0\Big(\frac{b_k}{e}\Big) \ge \\ \ge \sum\nolimits_{0\le k\le i-1}\kappa M_k\,\frac{1}{b_0^2+e^2}\,\frac{2\,e^2}{5\,b_0^3}\,\mathrm{r}_2^0\Big(\frac{b_0}{e}\Big) \tag{231}$$

(as $b_k \le b_0$ for $0 \le k \le i-1$). Then we obtain from (229) using (175) and (227) for $1 \le i \le I$ the chain of inequalities

$$\omega^2 > \frac{1}{b_i^2+e^2}\,\frac{2\,e^2}{5\,b_i^3}\,\mathrm{r}_2^0\Big(\frac{b_i}{e}\Big)\sum\nolimits_{i\le k\le I}\kappa M_k + \\ + \sum\nolimits_{0\le k\le i-1}\kappa M_k\,\frac{1}{b_k^2+e^2}\,\frac{2\,e^2}{5\,b_k^3}\,\mathrm{r}_2^0\Big(\frac{b_k}{e}\Big) > \\ > \frac{1}{b_0^2+e^2}\,\frac{2\,e^2}{5\,b_0^3}\,\mathrm{r}_2^0\Big(\frac{b_0}{e}\Big)\sum\nolimits_{i\le k\le I}\kappa M_k + \\ + \sum\nolimits_{0\le k\le i-1}\kappa M_k\,\frac{1}{b_0^2+e^2}\,\frac{2\,e^2}{5\,b_0^3}\,\mathrm{r}_2^0\Big(\frac{b_0}{e}\Big) = \\ = \frac{\kappa M}{b_0^2+e^2}\,\frac{2\,e^2}{5\,b_0^3}\,\mathrm{r}_2^0\Big(\frac{b_0}{e}\Big) = \omega^2\,, \tag{232}$$

that is impossible to satisfy. Thus, if there is an equilibrium for the rotating layered ellipsoidal body, then the body consists of the single layer ($I = 0$); in other words, it is a homogeneous ellipsoidal body.

This is an even more unexpected result, as the layered ellipsoidal body whose layer boundaries are confocal rotational ellipsoids is considered as a natural generalization of the layered spherical body (with homogeneous layers): in both cases the external gravitational field does not depend on the parameters of layers, but only on the total mass of the body (and on the excentricity of the surface of the body). Thus, if we wanted to create a model of the body generating the given external gravitational field (and ignoring the equilibrium), we could introduce as many layers with confocal ellipsoidal boundaries as desired; now we see that the requirement of an equilibrium makes such a model impossible.

Concerning the normal gravity field, we can say the following: even if we ignore the equilibrium conditions at the internal boundaries (at surfaces S_i for $1 \leq i \leq I$), there remain the two conditions at the surface of the body – (218) and (221) for $i = 0$. These conditions imply the formulae (225) and (227), and therefore there is a single possible value of angular velocity ω. Thus, there is no layered ellipsoidal body whose layer boundaries are rotational ellipsoids with common centre and rotation axis and whose layer density increases towards the centre, and such that the external gravitational field generated by this body is the normal gravitational field for the value of angular velocity ω different from those given by (227).

8. Discussion

It is evident that the results of the previous Section can be generalized to the case of infinitely many ellipsoidal layers and also for certain smooth density distributions. To be exact, consider the body composed of $I+1$ layers ($I \geq 0$) as follows: for each i ($0 \leq i \leq I$) let D_i be a domain whose boundary is a smooth, closed and simply connected surface S_i; the domains D_i have to satisfy the condition $D_{i+1} \subset D_i$ for each $0 \leq i < I$. Then the i-th layer is a domain defined as the set of points $\boldsymbol{r}$ such that $\boldsymbol{r} \in D_i$, $\boldsymbol{r} \notin D_{i+1} \cup S_{i+1}$; the domain D_0 is the interior of the body and S_0 is its surface. The density distribution $\rho(\boldsymbol{r})$ is positive in the whole interior of the body and it is defined as follows. For each i ($0 \leq i \leq I$), the interior limiting value of density at the surface S_i is a constant $\rho_{i,0}$: $[\rho(\boldsymbol{s})]_{\text{int}} = \rho_{i,0}$ for $\boldsymbol{s} \in S_i$. For each i ($0 \leq i < I$), the exterior limiting value of density at the surface S_{i+1} is a constant $\rho_{i,1}$: $[\rho(\boldsymbol{s})]_{\text{ext}} = \rho_{i,1}$ for $\boldsymbol{s} \in S_{i+1}$; for $i = I$ we define $\rho_{I,1}$ as the maximum of density in the domain D_I. The constants $\rho_{i,0}$ and $\rho_{i,1}$ have to satisfy the inequalities

$$0 \leq i \leq I : \quad \rho_{i,0} \leq \rho_{i,1} \, , \tag{233}$$

$$0 \le i < I: \quad \rho_{i,1} \le \rho_{i+1,0}\,. \tag{234}$$

For the i-th layer, if $\rho_{i,0} = \rho_{i,1}$, then the density $\rho(\boldsymbol{r})$ is in this layer equal to the constant $\rho_{i,0}$; if $\rho_{i,0} < \rho_{i,1}$, then for any ρ such that $\rho_{i,0} < \rho < \rho_{i,1}$ the set of points $\boldsymbol{r}$ for which $\rho(\boldsymbol{r}) = \rho$ is a smooth, closed and simply connected surface $S_i(\rho)$ which is the boundary of domain $D_i(\rho)$ and

$$0 \le i \le I, \quad \rho_{i,0} < \rho_0 < \rho_1 < \rho_{i,1}: \quad D_i(\rho_1) \subset D_i(\rho_0)\,. \tag{235}$$

Speaking freely, the density is in each layer either constant or it is continuously increasing towards the centre of the body. Finally, we impose the condition that the density distribution $\rho(\boldsymbol{r})$ is rotationally symmetric and symmetric with respect to the equatorial plane; the same thus holds for all above defined domains and surfaces.

Now we can formulate the generalization of the main result of the previous Section: for the layered body as defined above, if all surfaces S_i and $S_i(\rho)$ ($0 \le i \le I$, $\rho > 0$, the latter if they exist) are rotational ellipsoids with common centre and rotational axis, the layered body can be in equilibrium only if it is a homogeneous body. In addition, the external gravitational field of such layered body can be equal to the normal gravitational field only if the body is homogeneous and if the value of angular velocity ω is that corresponding to the homogeneous body.

The consequences of these results are far reaching, and therefore it is very surprising that even their mere existence is rarely mentioned in standard textbooks (the author itself has had no knowledge of them before finishing this work). It is astonishing that, as one can see from *Moritz (1990)*, Section 3.2.4, (theorem of Hamy – Pizzetti), these facts are known for more than 120 years and they were treated by several outstanding geodesists. We still note that our proof is completely different from that presented in *Moritz (1990)*.

On the other hand, all this does not mean that we do not know what can be the density distribution which generates the normal gravitational field. Quite on the contrary, there exists a (relatively simple) way to find not only some single density distribution, but all density distributions (from certain class of smooth functions) generating the (arbitrarily) given external gravitational field for a body with (almost arbitrary) shape with a smooth boundary. For the case of the body with the shape of the rotational ellipsoid this way is described in the author's work P99 (which itself is based on several previous author's works). Moreover, there is a straightforward generalization to the class of piecewise smooth density distributions. However, we shall not treat this matter here in order to keep the length of the present paper limited. We have to stress only the following extremely important aspect: the mentioned density distributions are constructed without considering any equilibrium condition. The question whether there can be found (in closed form) any density distribution satisfying also the equilibrium condition seems to be very difficult to answer.

We have to note that similar approach to the global inverse gravimetric problem for the body with the shape of the rotational ellipsoid was developed by H. Moritz; it is presented in *Moritz (1990)*, Chapters 5 – 7 (the author has had no previous knowledge of this book).

Another way out of the problem may be the direct calculation of the gravitational field of a homogeneous body whose shape differs slightly from the rotational ellipsoid (this does not mean that the difference is infinitesimal). Although it is unlikely that the result can be obtained in a closed form, there is a chance to express it as a series in terms of some parameter describing the deviation of the surface from the rotational ellipsoid. In our opinion, the expression of the gravitational intensity and potential in form of an integral over the surface of the body can represent the base for such approach. It seems that there is a chance to construct a layered body whose layer boundaries are surfaces of this kind and satisfy the equilibrium conditions.

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