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Two alternative techniques for fitting the gravimetric geoid for Egypt

Hussein ABD-ELMOTAAL, Atef MAKHLOOF*

Civil Engineering Department, Faculty of Engineering, Minia University, Minia 61111, Egypt; e-mails: Hussein.abdelmotaal@gmail.com, atefmakhloof@yahoo.com

Abstract: In this paper, two proposed geoid fitting techniques for Egypt's gravimetrically determined geoid and Global Positioning System GPS/levelling-derived geoid are introduced. First, any errors in the available GPS stations are ruled out. These methods rely on the absolute geoid difference, which is the gravimetric geoid height minus the geoid obtained through comparing GPS and levelling. The suggested geoid fitting techniques use an optimization algorithm scheme to choose the minimum number of the best-suited GPS stations to be used for fitting the gravimetric geoid. The least-squares collocation method is used to determine each GPS point's impact on the remaining GPS points. The GPS stations with the least impact on the other points are used for external validations, till an acceptable limit of the influence of the GPS points on the remaining ones (when the lowest standard deviation of the differences between gravimetric and geometric geoids is achieved with minimum average or standard deviation is larger than the accuracy of GPS observations, or standard deviation is larger than the target accuracy for geoid; or until a maximum of 30% of the GPS/levellings are excluded). This method operates in a way that automatically selects the fewest GPS stations that are most suitable for usage in the geoid fitting procedure. The geoid quality is then checked externally using the remaining GPS stations. A kriging trend surface is then taken out of the absolute geoid difference to complete the fitting. The proposed geoid fitting techniques are compared with that using polynomial regression of different degrees. The results proved that the proposed techniques give extremely better results. The findings of this study affirm that the adoption of the proposed techniques yields a geoid with an external accuracy of approximately 19 cm for Egypt.

Key words: Egypt, geoid fitting, GPS/levelling, internal and external check, polynomial regression

1. Introduction

Nowadays, most countries are seeking to acquire 1 cm or better geoid/quasi-geoid models covering their territories (*Smith et al., 2013; Farahani et al., 2017; Oršulić et al., 2020; Ellmann et al., 2020*). Therefore, the precise de-

*corresponding author, e-mail: atefmakhloof@yahoo.com

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termination of geoid models has begun to receive more attention in response to the widespread use of satellite-based positioning techniques, GPS (Global Positioning System), intending to replace geometric levelling measurements with GPS measurements during geodetic and surveying work.

The gravimetric determination of the geoid relies upon the solution of the spherical geodetic boundary-value problem and requires the evaluation of Stokes's surface convolutive integral. In practice, the gravimetric geoid is computed using a combination of terrestrial and satellite-derived gravity data. The approach taken in this contribution is to spectrally decompose the geoid height into the reference spheroid (low-frequency geoid) which is computed from a satellite-derived spherical harmonic global model and the high-frequency geoid which is computed from terrestrial gravity data. The high-frequency component of the geoid in this data combination requires the numerical evaluation of the adapted Stokes formula. Generally, its solution can be obtained by using discrete numerical integration (i.e. quadrature-based summation) or converting Stokes's convolutive integral from the space domain into a product of the spectra of Stokes's function with that of gravity data in the frequency domain and back again.

The Global Positioning System is a three-dimensional positioning system, which can naturally provide the three-dimensional Cartesian (and hence geodetic) coordinates at any point on the Earth's surface. Thus, the height (h) of a point above a selected reference ellipsoid can be determined at any point on the Earth. If orthometric height (H) is measured at a satellite position station by geometric levelling techniques, a direct and accurate measure of geoid undulation N can be obtained. Hence, the geoid undulation is simply the difference between the orthometric height and ellipsoidal height at the same point (*Nassar, 1984*). The use of GPS-derived undulations nowadays (GPS/levelling) permits an assessment of geoid undulation differences with relatively high accuracy.

In general, gravimetric geoids differ from geometric geoids for a variety of reasons. These include residual gravity anomalies, inaccuracies in the geopotential model's long wavelength, errors in the GPS ellipsoid height, and flaws in the spirit levelling. But, by modeling a bias and tilt to the inconsistencies, the error in the geometric geoid will be decreased (see e.g. *Eshagh and Sjöberg, 2008; Kiamehr and Eshagh, 2008; Eshagh, 2010; Eshagh and Ebadi, 2014; Eshagh and Zoghi, 2016* and *Eshagh and Berntsson, 2019*).

In reality, it is difficult to model the errors in the gravimetric and GPS-derived. Consequently, these errors lead to recurring differences between the geometric geoid generated by GPS or levelling and the gravimetric geoid. The gravimetric geoid may furthermore be subject to direct and indirect effects from the global geopotential model, which might lead to a systematic bias and/or tilt. Although these errors may be modeled by fitting the discrepancies with a model, there is no consensus on which model should be used.

Gravimetric geoid estimation from dense datasets, such as surface gravity, global geopotential models (GGMs), and topography, is commonly utilized on both regional and local scales (see e.g. *Denker et al., 2000; Smith and Roman, 2001*). The geoid is precisely recovered by the current models over Egypt at short wavelengths. But, inaccuracies in the truncation techniques and/or geopotential models (like EGG97 (*Denker et al., 2000*), JGEOID2000 (*Kuroishi, 2001*), and GEOID93) may affect this precision.

However, systematic errors in longer wavelengths may occur (*Milbert, 1995*). The orthometric heights acquired from GPS/levelling measurements, on the other hand, provide exact point-wise geoid undulations that comprise the entire gamut of geoid signals but do not provide the geoid heights strictly speaking. Consequently, they are extremely important to evaluate gravimetric geoid undulations. Gravimetric data and GPS/levelling geoid undulations should be merged to create a geoid model that is accurate and dependable in terms of spatial resolution (see e.g. *Smith and Milbert, 1999*). These characteristics make the study of geoid and GPS/levelling differences vital for both practical surveying and scientific applications. To this extent, several studies have been carried out in various locations (see e.g. *Forsberg and Madsen, 1990; Fotopoulos et al., 1999; Kearsley et al. 1993* and *Mainville et al., 1992*).

Fitting the gravimetric geoid to geodetic geoid at GPS/levelling points often involves a plane or low-order polynomial (*Featherstone et al., 1998*). For the model utilized in practical geodetic applications, such as large-scale map production, engineering projects, etc., the geometric method has traditionally been preferred. However, the geometrically derived geoid model's correctness is influenced by several variables. The distribution of reference stations (GPS/levelling Stations) must be as uniform as possible over the model area, and these sites must be selected to determine the likelihood that the geoid surface will change. Therefore, it would be advantageous to take the topographic features into account while selecting these reference locations. On the other hand, the number of reference points is stated as at least 1 point/20 km^2 for modeling the geoid. For this reason, the current research suggests two alternative techniques for fitting the geoid in Egypt as the number of available GPS/levelling stations is only thirty.

In many investigations, the geoid heights have been modeled using the polynomial surface fitting problem. The size of the research region and variance in geoid heights both affect the polynomial's degree. It is typical to use a low-order polynomial to simulate the typical fluctuations of a geoid surface in a relatively limited area of interest. However, alternative polynomial classifications, including bi-quadratic, bi-cubic, bi-quartic, and bi-quantic surfaces, are used for comparatively large or enormous areas (*Amiri-Simkooei et al., 2018*). To analyze and approximate the correctness of the geoid models in a restricted region, *Khazraei et al. (2017)* employed the bi-quadratic and bi-linear polynomials. In the study by *Das et al. (2018)*, the correcting surface had the highest accuracy on a nearly flat area while applying a third-degree polynomial. The five-degree bi-quantic polynomial was utilized by *Eteje et al. (2019).* The finite element-based bivariate (BIVAR) interpolation technique was used by *Erol and Erol (2021)* to simulate local geoid data. *Amiri-Simkooei et al. (2018)* introduced the LS-BICSA approach to estimate a 2D smooth surface of an unevenly distributed data set. The 2D bi-cubic spline approximation using least squares was implemented by *Hosseini-Asl et al. (2021).*

The ability to compute an accurate geoid model was made feasible by the possibility of fitting the gravimetric geoid model with GPS/levelling and by effectively employing a variety of data (*Erol, 2007; Kaloop et al., 2018, 2020, and 2022*). Finally, fitting gravimetric geoid or quasi-geoid models to GPS/levelling data is a technique now used in many countries. Many papers describe the fitting of the gravimetric geoid to GPS/levelling. An example of this is the fitting of the Australian Gravimetric Quasigeoid 2017 model (AGQG2017, *Featherstone et al., 2018a*) to a nationwide GPS/levelling dataset (*Featherstone et al., 2018b*) to provide a model of the separation between the Geocentric Datum of Australia 2020 ellipsoid and the Australian Height Datum, thus enabling a direct transformation between ellipsoidal and heights (cf. *Featherstone, 2000*).

The main aim of this study is to propose two alternative geoid fitting techniques for sparsely distributed GPS points, such as the situation in Egypt. The proposed geoid fitting techniques have been compared with the widely used in-practice, surface polynomial fitting technique.

2. Methodology and computation

In this study, the proposed geoid fitting techniques use an automated optimization scheme to select a number of the few best proper GPS stations to be used to fit the gravimetric geoid. The least-squares collocation technique is utilized to calculate each GPS point's impact on the other GPS points.

Using the least-squares collocation technique, the influence at each point is calculated from the neighboring points excluding the value at the computational point P. The equation of the least-squares collocation is expressed as (*Moritz, 1980; Fashir and Kadir, 1998; Tscherning 2002*):

$$\Delta N_p = \Big(C(p,p_1)\ C(p,p_2)\ ......\ C(p,p_n) \Big) \times \times \begin{pmatrix} C(p_1,p_1) & C(p_1,p_2) & ... & C(p_1,p_n) \\ C(p_2,p_1) & ... & ... & C(p_2,p_n) \\ ... & ... & ... & ... \\ C(p_n,p_1) & ... & ... & C(p_n,p_n) \end{pmatrix}^{-1} \begin{pmatrix} \Delta N_{p_1} \\ \Delta N_{p_2} \\ ... \\ \Delta N_{p_n} \end{pmatrix}, \qquad (1)$$

where $C(P,P_i)$ is the covariance between the point under consideration and the running nearby points and $C(P_i,P_j)$ represents the covariance between the two pairs of running nearby points.

In the current investigation, the generalized covariance model of Hirvonen has been identified and tested which is expressed as follows (*Moritz 1980, p. 179*):

$$C(P_i,P_j) = C(s) = \frac{C_0}{(1+A^2 s^2)^p}\,, \qquad (2)$$

where s refers to the distances between the pair of the considered points and the parameter A is presented by (*Abd-Elmotaal, 1992*):

$$A = \frac{1}{\xi}\left(2^{\frac{1}{P}} - 1\right)^{\frac{1}{2}}, \qquad (3)$$

with the empirical covariance function C_0, correlation length ξ. The parameter P depends on the geoid undulation type and a value of 0.25 has been used (*ibid.*).

Equation (2) demonstrates that the covariance function depends mainly on the inverse square distance, and it leads to severely ill conditional covariance matrices. Therefore, it has been replaced by the following local covariance function (*Fashir and Kadir, 1998, Eq. (3)*):

$$C(P_i, P_j) = C(s) = C_0 \left(1 + \frac{s}{R}\right)^{-1}, \tag{4}$$

where R refers to the mean radius of the Earth. The difference between the estimated values and the data values (residuals) is calculated and s refers to the distances between the pair of the considered points.

To apply the aforementioned technique, the absolute geoid differences (ΔN) for all GPS stations are computed:

$$\Delta N = N_G - N_{GPS}\,, \tag{5}$$

where N_G represents the gravimetric geoid height and N_{GPS} is the geoid height derived from GPS/levelling.

The total number of GPS stations is known to be N_P ($N_P = 30$ GPS stations are taken into consideration in the case of Egypt). The procedures below are used to calculate the impact of each GPS station on the other GPS stations knowing that the computations are performed on absolute geoid differences:

a) Assume it is required to compute the effect of point No. 1 (ΔN_1) on the other GPS stations (from 2 to 30 as in the case of Egypt).

b) Two different paths are used to determine the absolute geoid differences (ΔN_2) at point 2: the first path uses all values (absolute geoid differences) at all GPS points except the value at point 2 (Eq. (1)), while the second path uses all values at GPS stations except the values at points 1 and 2. Then, the difference between the two values resulting from the two paths is computed (ε).

c) The same computations in steps (b) are repeated for points $3, 4, ...$ to N_P (this means from 3 to 30).

d) As a result of steps (b) and (c), the differences at all points except point

1 are computed. Consequently, the standard deviation for all these differences is computed.

e) This standard deviation value in step (d) represents the impact of point 1 on the other GPS stations.

f) The above steps are applied for points $2, 3, \ldots\ldots, N_P$ respectively, and the standard deviation is generated for every point. Consequently, a file with standard deviations for all points is performed as shown in Table 2.

g) As the standard deviation is an important factor in picking the point with the least influence on the other GPS stations, the point with the lowest standard deviation (point 21) is chosen and deleted from the GPS file and then restored in another upset file (used for external validation). The flowchart for the above steps from (a) to (g) is shown in Fig. 1.

h) The updated GPS file (the number of GPS stations is reduced by one) is then processed using the same steps from (b) to (g) and another point is chosen and stored in the same file used for external checking.

i) The above step is repeated till this technique gives the best fitting.

j) The border crossing points (the four GPS/levelling corner points of the area under consideration) are not considered for external assessment.

The above steps summarize the *first technique for fitting* the gravimetric geoid to the geometric geoid (this technique depends on successive iterations).

The *second fitting technique* in this investigation depends on one solution (all the points having the minimum standard deviation and used for external checking are taken from the first iteration only).

Finally, the GPS points having the minimum influence on the remaining points are added to the subset of the GPS points till an acceptable limit of the influence of the GPS points on the remaining ones is achieved. For the current application of the proposed geoid fitting techniques, this acceptable limit has been set when the lowest standard deviation of the differences between gravimetric and geometric geoids is achieved with minimum average or standard deviation is larger than the accuracy of GPS observations; or standard deviations is larger than the target accuracy for geoid; or until a maximum of 30% of the GPS/levellings are excluded. As a result of this

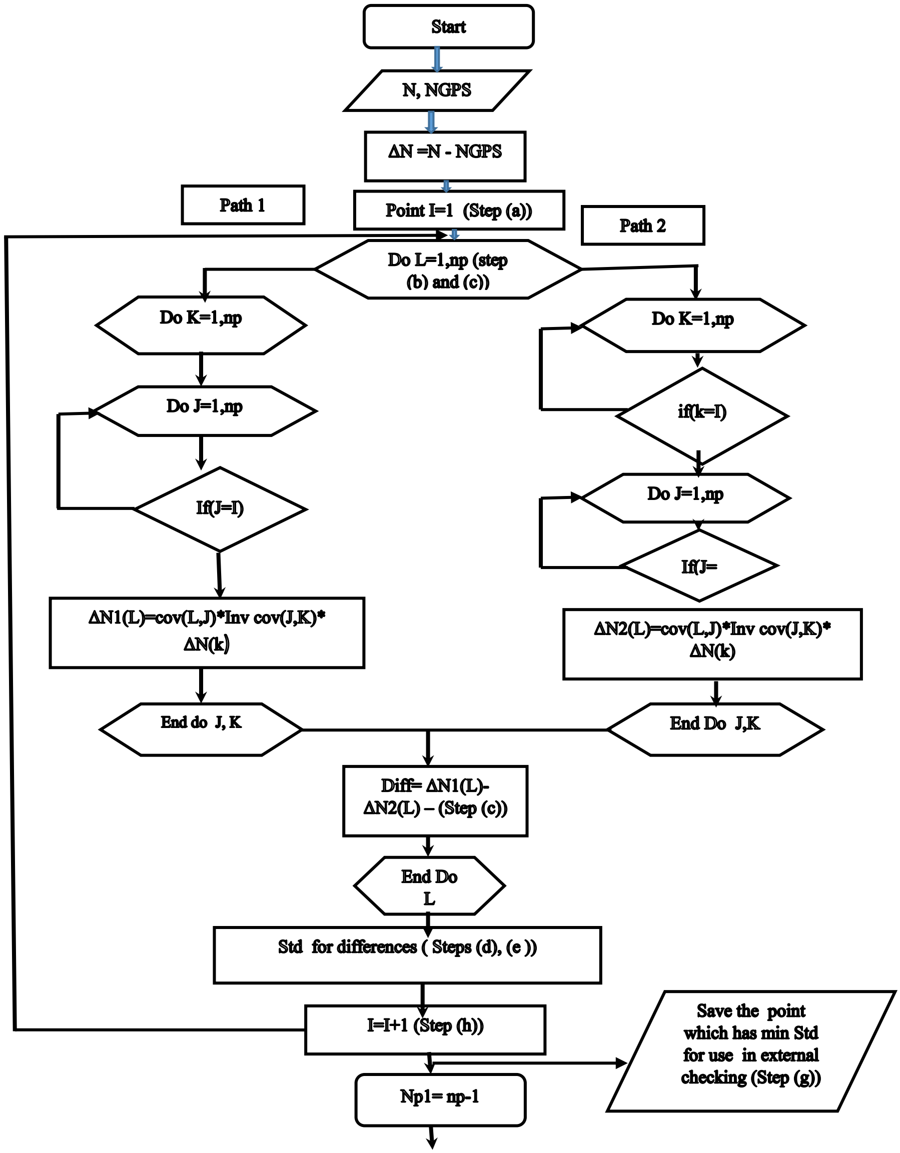


Fig. 1. Flow chart of the first iteration of the used methodology.

technique, two subsets are produced by the aforementioned technique, the first of which contains the stations with the least effect on the other GPS stations and is used to estimate the external check of the geoid quality. The other subset is utilized for the geoid fitting.

3. Data used

To validate the proposed techniques, two data sets are used: the first comprises 30 GPS data points with known geoid height in Egypt. These stations are regularly distributed all over the country (Fig. 2), however, the total number of GPS stations is small compared to Egypt's surface area.

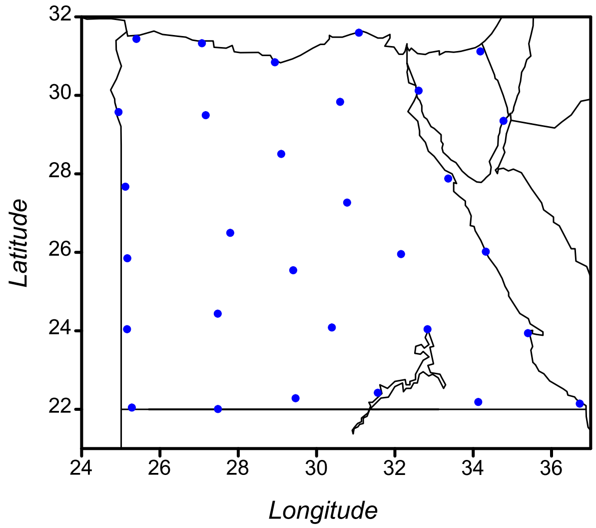


Fig. 2. Available GPS stations.

The second data set is the gravimetric geoid undulation for Egypt. The used gravimetric geoid is computed using a high-degree tailored geopotential model for Egypt (after *Abd-Elmotaal, 2008*) see Fig. 3. In this model, a high-degree reference geopotential model customized for Egypt is created. To estimate the harmonic coefficients of the high-degree tailored reference model using a Fast Fourier Transform technique and an iterative process, the local and global data sets for isostatic gravity anomalies are combined.

This process improves the accuracy of the calculated harmonic coefficients and reduces the residual field. A modified geopotential model relating to the free-air field has been created by restoring the impact of the topographic-isostatic masses that had previously been eliminated. The lower harmonics up to degree 20 have been locked at their values as per the EGM2008

Table 1. Absolute geoid differences for Egypt, units in metres.

Point No.	latitude	longitude	Absolute geoid difference
1	22.14	36.72	−0.71
2	22.18	34.12	1.21
3	22.42	31.56	−0.08
4	22.28	29.46	0.07
5	24.43	27.47	0.97
6	26.49	27.79	5.69
7	24.08	30.39	1.17
8	24.04	32.83	−2.48
9	23.94	35.39	−2.21
10	26.01	34.32	−2.18
11	25.95	32.15	−3.22
12	25.54	29.40	0.73
13	28.50	29.09	1.68
14	27.26	30.77	−1.05
15	27.88	33.36	−4.42
16	29.35	34.77	−6.12
17	31.11	34.18	−0.07
18	30.11	32.60	−2.22
19	31.59	31.08	−0.29
20	39.83	30.60	−0.56
21	30.84	29.93	1.11
22	31.32	27.07	1.20
23	31.43	25.39	0.80
24	29.57	24.94	1.41
25	29.49	27.16	4.71
26	27.67	25.11	6.00
27	25.84	25.16	4.45
28	24.04	25.16	0.61
29	22.00	27.48	−1.71
30	22.04	25.28	−1.99

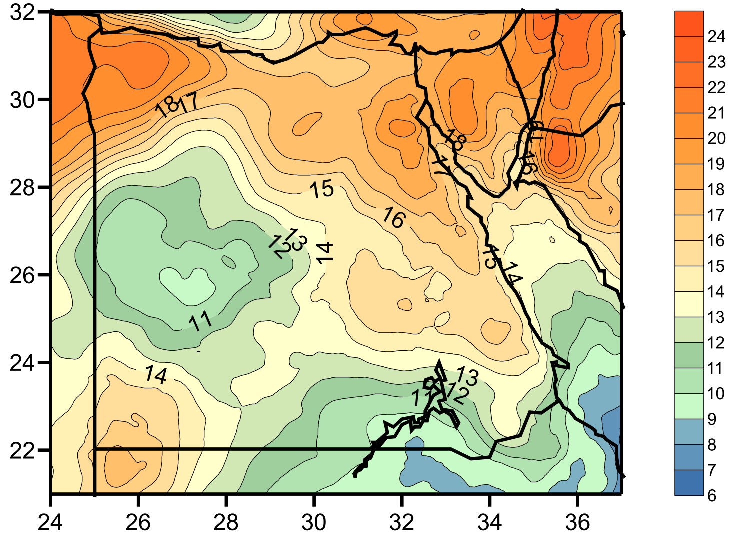


Fig. 3. Gravimetric geoid for Egypt after *Abd-Elmotaal (2008)*.

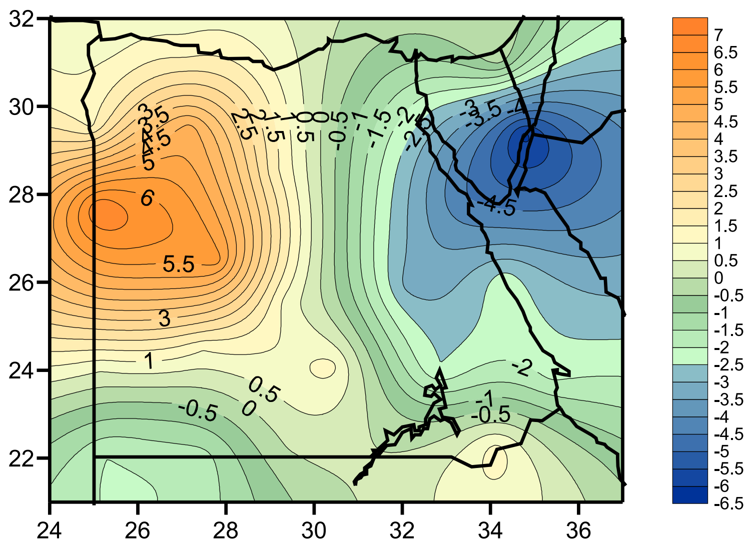


Fig. 4. Absolute geoid differences ($N_G - N_{GPS}$).

geopotential model because the low order harmonics of the anomalous gravitational potential are, to a large part, caused by the density disturbances in the upper mantle and even deeper sources. Then, using the remove-restore approach, a gravimetric geoid for Egypt was calculated using this customized geopotential model.

Our computations are based on the absolute geoid differences between the two surfaces. The absolute geoid differences are shown in Fig. 4. The absolute geoid differences for GPS stations in Egypt are shown in Table. 1.

The values of absolute geoid differences are relatively big for several reasons:

1. Locally linked inaccuracies in the geoid/quasi-geoid are caused by systematic errors in gravity measurements, mistakes, and lack of data.
2. Mistakes and errors inside the GPS geodetic network.
3. The impact of the troposphere on GPS readings creates geoid height inaccuracies that are locally connected.
4. The errors in gravity observations used in the gravimetric technique of the geoid.
5. The error in the global geopotential model used.
6. The errors in the residual gravimetry used as a result of using constant density in terrain effect computations.
7. The substantial data gaps of gravity for the area under consideration.
8. The numerical integration of the Stokes function leads to inaccurate results for geoid.

4. Numerical computations

Two methods are used in this investigation. The thirty accessible GPS stations in Egypt have been subjected to the steps mentioned in Section 2. Table 2 displays the results of the first iteration.

Table 2 demonstrates that point No. 21 has the minimum impact, hence it is taken from the points as a result and stored in a new file to use for external check. Consequently, there are 29 points left. The results of the second iteration are shown in Table. 3.

Table 3 demonstrates that point No. 20 has the minimum impact, hence it is taken from the GPS stations as a result and added to the file used for the external check. Consequently, there are 28 points left. The above procedures are applied iteratively. After applying many iterations, the results for eight iterations (the points used for the external checks in case of fitting the gravimetric geoid to the geometric geoid) are shown in Fig. 5.

Table 2. Results of the first iteration.

Point No.	Latitude	longitude	Std.
1	22.14	36.72	0.09
2	22.18	34.12	0.31
3	22.42	31.56	0.04
4	22.28	29.46	0.04
5	24.43	27.47	0.07
6	26.49	27.79	0.24
7	24.08	30.39	0.17
8	24.04	32.83	0.16
9	23.94	35.39	0.13
10	26.01	34.32	0.17
11	25.95	32.15	0.14
12	25.54	29.40	0.08
13	28.50	29.09	0.05
14	27.26	30.77	0.03
15	27.88	33.36	0.14
16	29.35	34.77	0.38
17	31.11	34.18	0.32
18	30.11	32.60	0.07
19	31.59	31.08	0.05 (Border point)
20	29.83	30.60	0.03
21	30.84	28.93	0.02
22	31.32	27.07	0.08
23	31.43	25.39	0.15
24	29.57	24.94	0.34
25	29.49	27.16	0.17
26	27.67	25.11	0.35
27	25.84	25.16	0.15
28	24.04	25.16	0.14
29	22.00	27.48	0.19
30	22.04	25.28	0.21

Then, the internal check (residuals at the GPS stations used for the geoid fitting) of the first technique is illustrated in Table 4, where the statistics for the absolute geoid difference residuals are shown. The internal points used for fitting the gravimetric geoid are checked for different numbers of stations.

Table 4 shows that the internal precision (standard deviation) does not

Table 3. Results of the second iteration.

Point No.	latitude	longitude	Std.
1	22.14	36.72	0.09
2	22.18	34.12	0.32
3	22.422	31.56	0.04
4	22.28	29.46	0.04
5	24.43	27.47	0.07
6	26.49	27.79	0.24
7	24.08	30.39	0.17
8	24.04	32.83	0.17
9	23.94	35.39	0.13
10	26.01	34.32	0.17
11	25.95	32.15	0.14
12	25.54	29.40	0.08
13	28.50	29.09	0.06
14	27.26	30.77	0.04
15	27.88	33.36	0.15
16	29.35	34.77	0.39
17	31.11	34.18	0.33
18	30.11	32.60	0.07
19	31.59	31.08	0.04
20	29.83	30.60	0.02
21	31.32	27.07	0.09
22	31.44	25.39	0.15
23	29.57	24.94	0.35
24	29.49	27.16	0.18
25	27.67	25.11	0.36
26	25.84	25.16	0.15
27	24.04	25.16	0.13
28	22.00	27.48	0.19
29	22.04	25.28	0.21

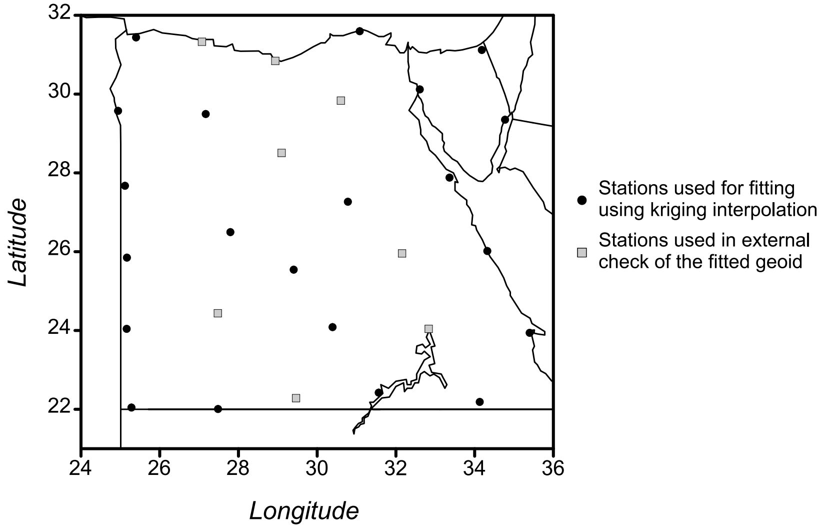


Fig. 5. The two sets of GPS stations used for geoid fitting and external check (after eight iterations).

Table 4. Absolute geoid difference for internal checks for the first technique.

Number of GPS stations used for geoid fitting	Min [m]	Max [m]	Avg [m]	St. dev. [m]
27 stations	−0.05	0.06	0.0012	0.03
26 stations	−0.05	0.06	0.0014	0.03
25 stations	−0.05	0.06	−0.0015	0.03
24 stations	−0.05	0.06	−0.0015	0.03
23 stations	−0.05	0.06	−0.0015	0.03
22 stations	−0.05	0.06	−0.0016	0.03
20 stations	−0.05	0.06	−0.0016	0.03
18 stations	−0.08	−0.08	−0.0016	0.03

depend on the number of the used GPS stations for geoid fitting. As in the fitting process, all 22 points are used.

Table 5 provides statistics for the remaining residuals at the GPS stations that were not used for the geoid fitting. This process presents the external check which indicates the geoid fitting quality. The rows of Table 4 and row No. 6 of Table 5 are correspondent (the sum is the total number of GPS stations, i.e., 30).

Table 5. Absolute geoid difference for external checks for the first technique.

Number of check stations	Min [m]	Max [m]	Avg [m]	St. dev. [m]
3 stations	−0.40	0.10	−0.26	0.15
4 stations	−0.67	−0.31	−0.43	0.16
5 stations	−0.67	−0.15	−0.38	0.19
6 stations	−0.67	0.29	−0.25	0.23
7 stations	−0.67	0.31	−0.29	0.33
8 stations	−0.67	0.31	−0.25	0.33
10 stations	−1.04	0.31	−0.43	0.36
12 stations	−1.53	0.21	−0.72	0.54

Table 5 shows that the best geoid fitting, expressed by the minimum standard deviation of the residual and minimum average occurs where 22 GPS stations have been used for fitting the gravimetric geoid and only eight GPS stations were used for the external check as the standard deviation of the differences is the smallest and the average is the minimum.

To save time, only the first iteration is tested for *the second proposed approach* in this study. This means that all points used for the external check are taken from the first iteration (Fig. 1 and Table 2). The external check

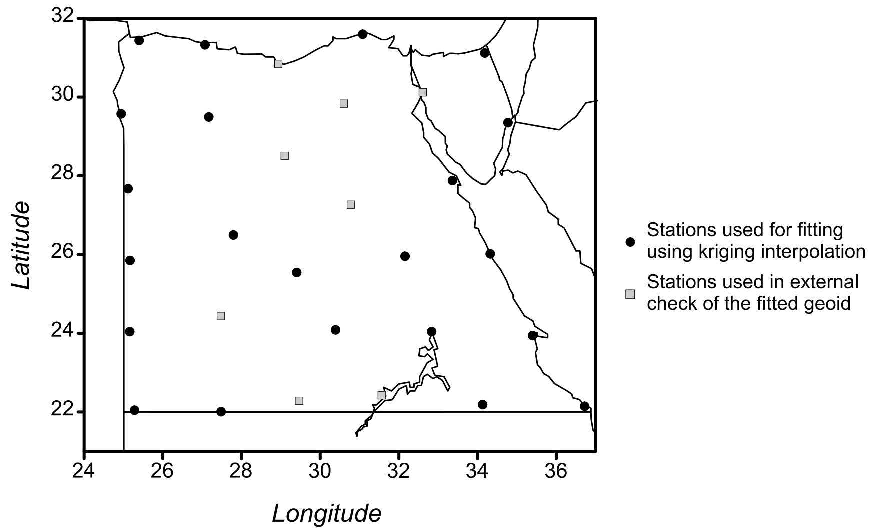


Fig. 6. The two sets of GPS stations used for geoid fitting and external check (only one iteration as in Table 1).

of the geoid fitting quilty is performed using the GPS stations with the least effect on the other stations (i.e., with the lowest residual standard deviation and minimum average). For the geoid fitting procedure, the remaining GPS points are utilized. The results of this iteration have been taken from Table 1 and shown in Fig. 6.

The internal check of the second technique is illustrated in Table 6, where the statistics of the residuals of the absolute geoid differences are shown. Of course, the internal precision does not depend on the number of the used GPS stations for the geoid fitting.

Table 6. Absolute geoid difference for internal checks for the second technique.

Number of check points	Min [m]	Max [m]	Avg [m]	St. dev. [m]
27 stations	−0.05	0.06	0.0012	0.03
26 stations	−0.05	0.06	0.0014	0.03
25 stations	−0.05	0.06	−0.0015	0.03
24 stations	−0.05	0.06	−0.0015	0.03
23 stations	−0.05	0.06	−0.0015	0.03
22 stations	−0.05	0.06	−0.0016	0.03

Comparing Tables 4 and 6 shows that the inter-checks remain the same for the two techniques.

Table 7 shows the external check for the second GPS fitting technique. It shows that the best geoid fitting expressed by the minimum standard deviation of the residual and minimum average occurs where 26 GPS stations have been used for the fitting and only 4 GPS have been used for the external check.

For the sake of comparison, the fitting of the gravimetric geoids within the current study was also done by subtracting a polynomial regression sur-

Table 7. Absolute geoid difference for external checks for the second technique.

Number of check stations	Min [m]	Max [m]	Avg [m]	St. dev. [m]
3 stations	−0.17	0.30	0.01	0.28
4 stations	−0.18	0.31	0.01	0.22
5 stations	−0.94	0.32	−0.37	0.36
6 stations	−0.94	0.30	−0.31	0.42
7 stations	−0.92	0.22	−0.40	0.42
8 stations	−1.42	0.19	−0.67	0.60

face of first, second, and third orders using the software SURFER. Table. 8 lists the residuals for the 30 GPS points after the polynomial surface subtraction. Given that all GPS stations were employed in the fitting strategies, Table 8 indicates the fitted geoids' internal precision.

Table 8. Geoid fitting using polynomial regression (using SURFER).

Polynomial order	Min [m]	Max [m]	Avg [m]	St. dev. [m]
1st order	−4.29	4.50	0.00	2.24
2nd order	−2.89	4.19	0.00	1.99
3rd order	−2.23	2.41	0.00	1.23

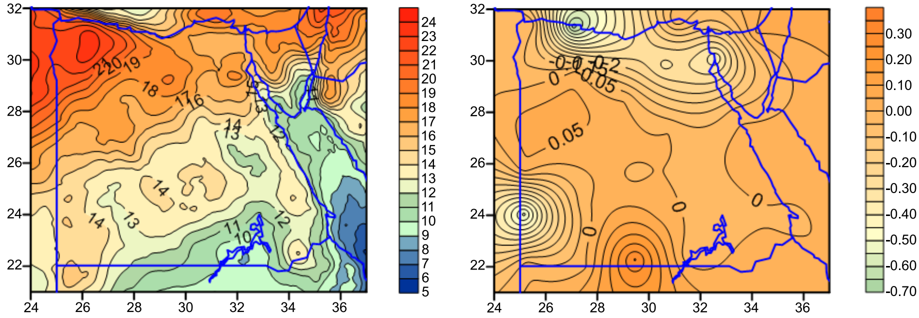
Comparing Tables 5, 7, and 8 illustrates that the suggested two geoid fitting techniques in this paper are more accurate than the polynomial regression fitting technique till the third order. It can be concluded that using four or five-order polynomial regression gives small residuals at 30 GPS stations. However, these polynomials are not suitable for the case in Egypt as the number of available GPS/levelling stations is only thirty which is not comparable with the area of the country. This is the main reason for our study.

Finally, Figures 7 and 8 show the fitted geoid (A) and the absolute geoid difference (B) using the two proposed geoid fitting techniques with the best number of GPS stations used for the geoid fitting (8 and 4 respectively). Figures 7 and 8 confirm again that the second proposed geoid fitting technique gives better geoid quality.

5. Conclusion

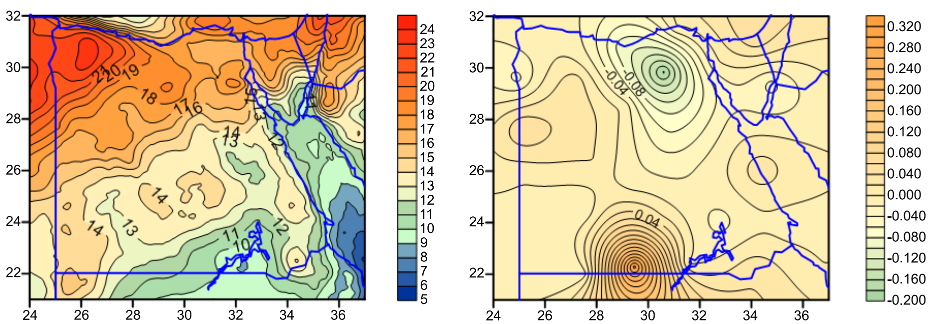
Two powerful alternative geoid fitting techniques have been proposed in the current investigation. They have been successfully applied to fit the gravimetric geoid for Egypt. The quilty of the fitted geoid using the proposed geoid fitting techniques expressed by the residual at the external checkpoints is one and a half decimeters. This quality is relatively too good compared to the very limited number of available GPS stations in the country.

For the sake of comparison, the gravimetric geoid for Egypt has been fitted to the GPS/levelling-derived geoid using surface polynomials regression fitting techniques. In this case, about 75% of the available GPS stations



A **B**

Fig. 7. Final geoid after using 8 GPS stations for the external checking (in the case of the first technique).



A **B**

Fig. 8. Final geoid after using 26 GPS stations for the case of the second technique.

have been used to fit the gravimetric geoid and the remaining 25% of GPS stations have been used to estimate the quality of the fitted geoid. The results proved that the proposed geoid fitting techniques within the current investigation are superior.

The implanted tests of the proposed geoid fitting techniques illustrate their capability to fit the gravimetric geoid in the case of sparse GPS stations. The proposed techniques need to be tested in case of dense GPS station coverage to determine their suitability.

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