Analytical model of the surface displacement and gravity changes due to the point source of the heat in the viscoelastic halfspace with topography

L. Brimich
Geophysical Institute of the Slovak Academy of Sciences[1]

Abstract: In the paper a method for including topographic effects in a thermo-viscoelastic model was described. An approximate methodology for the consideration of topography in the computation of thermo-viscoelastic displacement and gravity changes was used. On that way we allow to obtain a relatively general and simple solution useful for solving the inverse problem.

Key words: thermoviscoelastic deformation, effect of topography, gravity changes

1. Introduction

A magma intrusion in the Earth's crust will cause effects (for example deformation and gravity changes) related to its mass as well to the pressurization of the chamber due to overfilling or temperature changes. The source of deformation is hypothesized as a hydrostatic pressure source, embedded in a homogeneous elastic half-space. The pressure source is considered as a strain nucleus, i.e., a point like a source with radial expansion, that is similar to the inflation of a spherical cavity. A finite source can be satisfactorily approximated by a point source, provided that the source dimensions are small with respect to source depth.

The theory of the thermoelastic phenomena shows that thermo-elastic stresses and deformations can arise in an elastic continuum if an inhomogeneous temperature field exists in the media (see e. g., *Nowacki, 1962*).

[1] Dúbravská cesta 9, 845 28 Bratislava, Slovak Republic; e-mail: geofbrim@savba.sk

That is the reason, *Hvoždara and Rosa (1979, 1980)* carried out a theoretical analysis of thermo-elastic deformations of a homogeneous half-space due to a point or linear source of heat, located at a particular depth in the half-space. They proved that thermo-elastic stresses are on expansive type and that they considerably disturb the normal lithostatic stress, specially near the surface of the half-space. *Hvoždara and Brimich (1991)* presented basic formulae and the results of numerical calculations for the simplified mathematical models of two important effects due to magmatic bodies in the Earth's lithosphere: a) static thermoelastic deformations, b) static elastic deformations due to upward pressure. The thermo-viscoelastic deformation field due to a source of heat of prismatic shape embedded in a viscoelastic half-space and the formulae for the gravity changes due to the volume dilatation connected with the deformation field are derived in *Brimich (2000). Folch et al. (2000)* obtained and compared analytical and numerical solutions for ground displacement caused by an overpressurized magma chamber placed in a linear viscoelastic media composed by a layer over a half-space. Different parameters such as size, depth or shape of the chamber, crustal rheologies or topography are considered and discussed. The effect of the topography is also considered. *Fernández et al. (2001)* presented a method extension of a deformation model previously developed to compute effects due to volcanic loading in elastic-gravitational layered media (*Rundle, 1982, 1983; Fernández and Rundle, 1994; Fernández et al., 1997)*, for the computation of time-dependent deformation, potential and gravity changes due to magmatic intrusions in a layered viscoelastic medium. They assumed a plane Earth geometry consists of welded elastic and viscoelastic layers overlying a viscoelastic half-space. They found that, in line with prior results obtained by other authors (see e.g., *Bonafede et al., 1986*), introducing viscoelastic properties in all or part of the medium can extend the displacements and gravity changes considerably, and therefore lower pressure increases are required to model given observed effects. The approximation of Earth's surface as flat and use half-space solutions can lead to erroneous interpretation of the deformation data (see e.g., *Cayol and Cornet, 1998; Williams and Wadge, 1998, 2000; Folch et al., 2000). Williams and Wadge (1998, 2000)* and *Cayol and Cornet (1998)* pointed that topography has a significant effect on predicted surface deformation by elastic models in regions of significant relief. We used the methodology described in *Williams and Wadge (1998)* to introduce the topographic effects in the thermo-elastic and thermo-viscoelastic models we still get analytical solutions. The advantage of this assumption is something very clear, it allows to obtain a relatively general and simple solution useful for solving the inverse problem (see e.g., *Michalewicz, 1994; Yu, 1995; Yu et al., 1998; Tiampo et al., 2000*).

2. Thermo-viscoelastic deformation model

Elastic and thermoelastic models have allowed an explanation of the measured geodetic data in many volcanic regions, particularly when movements occur on relatively short timescales. The time evolution of heating of the halfspace (lithosphere) and associated deformation with it can be mathematically calculated by means of the theory of thermo-viscoelastic deformation. We consider a non-steady point source of heat located at depth ζ in the viscoelastic halfspace $z > 0$. For the uncoupled thermoviscoelastic problem, the temperature disturbance field $T(x, y, z, t)$ due to this source must obey the equation (*Nowacki, 1962*):

$$\lambda_T \Delta^2 T + w\delta(x)\delta(y)\delta(z-\zeta)H(t) = c_p \rho \frac{\partial T}{\partial t} \quad , \tag{1}$$

where λ_T is heat conductivity, c_p is specific heat under constant pressure, ρ is the material density, w is the power of heat source, δ is the Dirac function, $H(t)$ is Heavside's unit step function:
$H(t) = 0$ for $t < 0$,
$H(t) = 1$ for $t > 0$.
If the surface of the halfspace is kept at a constant temperature, which can be taken to be zero, then we have the boundary condition on the surface $z = 0$:

$$T(x, y, z, t)|_{z=0} = 0 \quad . \tag{2}$$

Considering the initial temperature disturbance in all points of the halfspace as zero, we obtain the initial condition for $t = 0$:

$$T(x, y, z, t)|_{t=0} = 0 \quad . \tag{3}$$

Then, the solution of Eq. (1) under the boundary and initial conditions, is obtained in the form (*Carslaw and Jaeger, 1959*):

$$T(r,z,t) = \frac{w}{4\pi\lambda_T}\left\{R_1^{-1} erfc\left(\frac{R_1}{\sqrt{4\kappa t}}\right) - R_2^{-1} erfc\left(\frac{R_2}{\sqrt{4\kappa t}}\right)\right\} \quad , \tag{4}$$

where $R_1 = [r^2 + (z-\zeta)^2]^{1/2}$; $R_2 = [r^2 + (z+\zeta)^2]^{1/2}$, with $r = (x^2+y^2)^{1/2}$ being the horizontal distance from the polar axis z and $\kappa = \lambda_T/(c_p\rho)$. The complementary error function *erfc(s)* is defined by:

$$erfc(s) = 1 - \frac{2}{\sqrt{\pi}}\int\limits_0^s e^{-u^2} du \quad . \tag{5}$$

The time and space variable temperature disturbance causes variable stresses and displacements. Since the process of temperature change is much slower in comparison with the propagation time of elastic waves, it is sufficient to consider the static equilibrium equation for a viscoelastic body:

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad i = 1,2,3 \quad , \tag{6}$$

where σ_{ij} is the viscoelastic stress tensor. In the purely elastic case the components σ_{ij} are given by the Duhamel-Neumann relation, but in the viscoelastic case the stress-strain relations are given by more complicated formulae (*Nowacki, 1962*).

In order to obtain the actual temporal behaviour of the displacements and stresses we have to Laplace transform these quantities. The detailed calculation was performed in *Hvoždara (1992)*.

The calculation was performed for a Kelvin body, for which the generalized Duhamel-Neumann relation has the form:

$$\sigma_{ij}(x_r,t) = 2\mu\left(1 + t^*\frac{\partial}{\partial t}\right) e_{ij}(x_r,t) + \delta_{ij}\left\{\frac{1}{3}\left[3K - 2\mu\left(1 + t^*\frac{\partial}{\partial t}\right)\Theta(x_r,t)\right] - 3K\alpha_T T(x_r,t)\right\}, \tag{7}$$

where e_{ij} is the strain tensor, μ is the modulus of rigidity (the Lamé constant), $K = \lambda + 2\mu/3$ is the bulk modulus, $t^* = \eta/\mu$ is decay time, η being the viscosity of material, α_T is the thermal coefficient of the linear expansion and $\Theta(x_r, t)$ is dilatation.

For the time dependance of displacements u and stresses σ on the surface of the viscoelastic half-space we have the following formulae (*Hvoždara, 1992*):

$$u_r(r,0,t) = \frac{Qr}{\pi}\int\limits_0^t V(t-\tau)S_1(r,\tau)\,d\tau,$$

$$u_z(r,0,t) = -\frac{Q}{2\pi}\left\{\zeta R_0^{-3}b(t) + \frac{2\zeta}{\sqrt{\pi}}\int\limits_0^t b(t-\tau)\tau^{-1}(4\kappa\tau)^{\frac{-3}{2}}e^{\left(\frac{-R_0^2}{4\kappa\tau}\right)}d\tau - \right.$$
$$\left. -\int\limits_0^t W(t-\tau)S_2(r,\tau)d\tau\right\}, \tag{8}$$

$$\sigma_{rr}(r,0,t) = \frac{2Q}{\pi}\left\{\int\limits_0^t B(t-\tau)S_0(r,\tau)\,d\tau - \int\limits_0^t U(t-\tau)S_1(r,\tau)\,d\tau\right\},$$

$$\sigma_{\varphi\varphi}(r,0,t) = \frac{2Q}{\pi}\left\{\int\limits_0^t N(t-\tau)S_0(r,\tau)\,d\tau + \int\limits_0^t U(t-\tau)S_1(r,\tau)\,d\tau\right\}.$$

The meaning of the symbols in Eqs (8) is detailed described in (*Hvoždara, 1992*).

In addition we can calculate the perturbation of gravity due to a point source of heat. There are two principal reasons for the gravity changes. The first one is the change of density ρ_0 by the increment of $\Delta\rho$ due to volumetric dilatation. The density change $\Delta\rho$ generates perturbation of the gravity potential, which for $z \geq 0$ obeys the Poisson equation. For $z < 0$ this potential satisfies Laplace equation, i.e., it's an harmonic function. *Brimich (1998)* obtained the following formula for the gravity anomaly:

$$\Delta g_{TVE}(r,0,t) = \frac{1}{2}G\rho_0 Q\left\{W(t)\frac{-\zeta}{R_0^3} + 2\zeta\int\limits_0^t W_2(t-\tau)\frac{e^{-R_0^2/(4\kappa\tau)}}{(2\kappa\tau)\sqrt{4\pi\kappa\tau^3}}d\tau\right\}, \tag{9}$$

where $G = 6.67 \times 10^{11}\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$ is gravity constant and the meaning of the other symbols in Eq. (9) is detailed described in *Brimich (1998)* . The second reason for the gravity changes is the free-air change and Bouguer correction as an effect of vertical uplift of the surface above the source of heat. The gravity effect due to the upward doming of the surface of the Earth, that originally was the plane $z = 0$, is given by the sum of the free-air change of gravity and the Bouguer correction:

$$\Delta g_{FAB} = [-\gamma_{FA} + 2\pi G\rho_0]h(r), \tag{10}$$

where $h(r) = -u_z(0, r, t)$ is the doming, i.e., the vertical uplift, and $\gamma_{FA} = 3.086 \times 10^{-6}\ \mathrm{ms}^{-2}/\mathrm{m}$ is the vertical gradient of normal gravity and $2\pi\, G\rho_0$ is the Bouguer correction.

3. The effect of topography

In this chapter the effect of the topography on the surface displacements and gravity changes obtained by the thermo-viscoelastic model described above is investigated. We propose a simple method of evaluating the topographic effects in three-dimensional deformation model that consists of assuming a different source depth at each point for which a solution is desired. This methodology was introduced by *Williams and Wadge (1998)* and permits that we still have analytical solutions even if we relax the restriction of a free flat surface. The analytical solutions are useful for solving of the inverse problem and avoid to include numerical models that can be time consuming. Therefore, we allow magma chamber depth to vary with topography, thus in the equations (8), (9) and (10) ζ is replaced by $\zeta' = \zeta + H$, where H is the point elevation we want to obtain the viscoelastic deformation and gravity changes. If the topographic effect is due primarily to the distance of the free surface from magma chamber rather than the local shape of the free surface, this methodology comes near the actual case (*Williams and Wadge, 1998, 2000).* To study the effect of the topography the relief of an area can be represented by a volcanic cone with height H and average slope of the flanks α. We consider the surface displacements and respective gravity changes caused by a point source of heat located beneath an axisymmetrical volcano with average slopes of their flanks of

0°, 15°, 20°, and 30°. The volcano models with slopes of 15° and 20° are representative of basaltic shield volcanoes, whereas the volcano models with slopes 30° are representative of andesitic volcanoes. Schematic illustration of the problem is given in Fig. 1. The effect of the topography is neglected when $\alpha = 0 (H = 0)$. The rheological behaviour of the crust is represented by a homogeneous halfspace Kelvin's type with Lamé parameters λ and μ with the topography characterized by the same parameters. As a reference model we have used a point source of heat at the depth $\zeta = 2$ km, its intensity (power) $w = 2.6384 \times 10^7$W in order to achieve the epicentral heat flow anomaly $q_z(0) = 42$ mW/m^2, since $\mathrm{q_z}(0) = w(2\pi\zeta^2)^{-1}$. The density and elastic parameters of the halfspace and the thermal parameters of the medium are following:

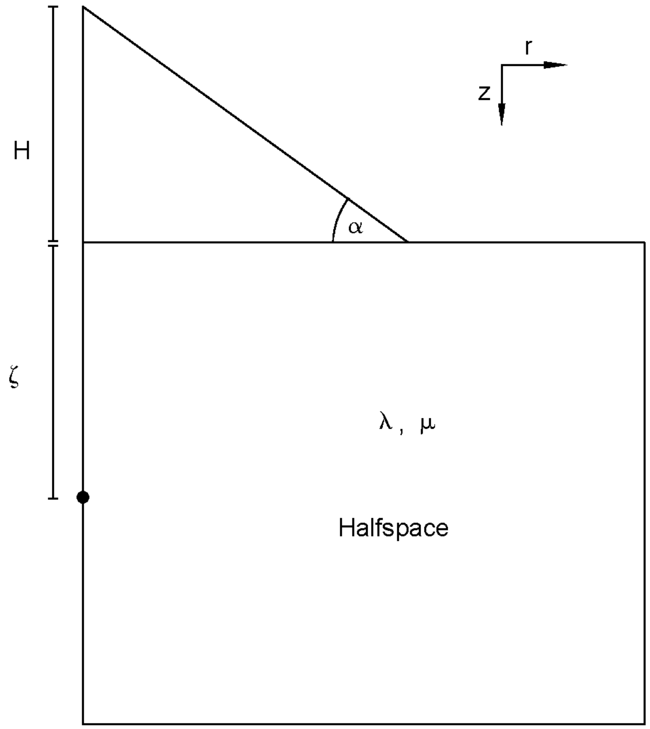

Fig. 1. Characteristics of the model used to determine influence of the topography on the surface displacements.

$\lambda = 7.05 \times 10^{10}$ Pa
$\mu = 6.075 \times 10^{10}$ Pa
K = $1.11{\times}10^{10}$ Pa
$\nu = 0.26857$
$\rho = 3000$ kgm^{-3}
$\lambda_T = 3$ Wm$^{-1}K^{-1}$
$c_p = 840$ Jkg$^{-1}K^{-1}$
$\alpha = 10^{-6}K^{-1}$
We set the decay time for the Kelvin's type of the viscoelastic body as $t^* = 3.3 \times 10^{12}$s, where $t^* = \eta/\mu$ and η is the mean viscosity of the crustal rocks. The results for the depth $\zeta = 2$ km ($t_\kappa = 8.37 \times 10^9$ s) are presented in Figs 2 to 4. The results are compared with the flat-surface solution given by the analytical method. The curves for $t/t_\kappa = 0.5, 1.0, 2.0, 3.0, 5.0, 7.0$ gradually approach the curves that were calculated by means of the formulae for the stationary thermoelastic problem (*Hvoždara and Brimich, 1991*). We can see that the displacements and gravitational anomalies approach their static values slowly, because of the viscoelastic behaviour of the half-space, which is mathematically expressed by the convolution integrals in the previous chapter. As it is pointed out by others authors, the principal effect of topography is a reduction of vertical displacement and total gravity anomaly magnitudes due to a the greater distance from the source to the free surface (the steeper the volcano, the flatter displacement field and gravity change). Fig. 4 shows that the topography effect changes the pattern of the total gravity anomaly, too. Vertical displacements and total gravity changes are mostly influenced by the topographic effect thus neglecting the topography may lead to a mis-interpretation of the volume change of the source. We observe in our results that, as *Folch et al. (2000)*, the effects of the topography are dramatically emphasized in the viscoelastic case.

4. Conclusions

The results show that the thermo-viscoelastic solution gradually approach the solution got for the stationary problem (thermo-elastic solution). The models used to interpret the geodetic data measured in volcanic areas, typically compute the deformation field and gravity changes at the surface

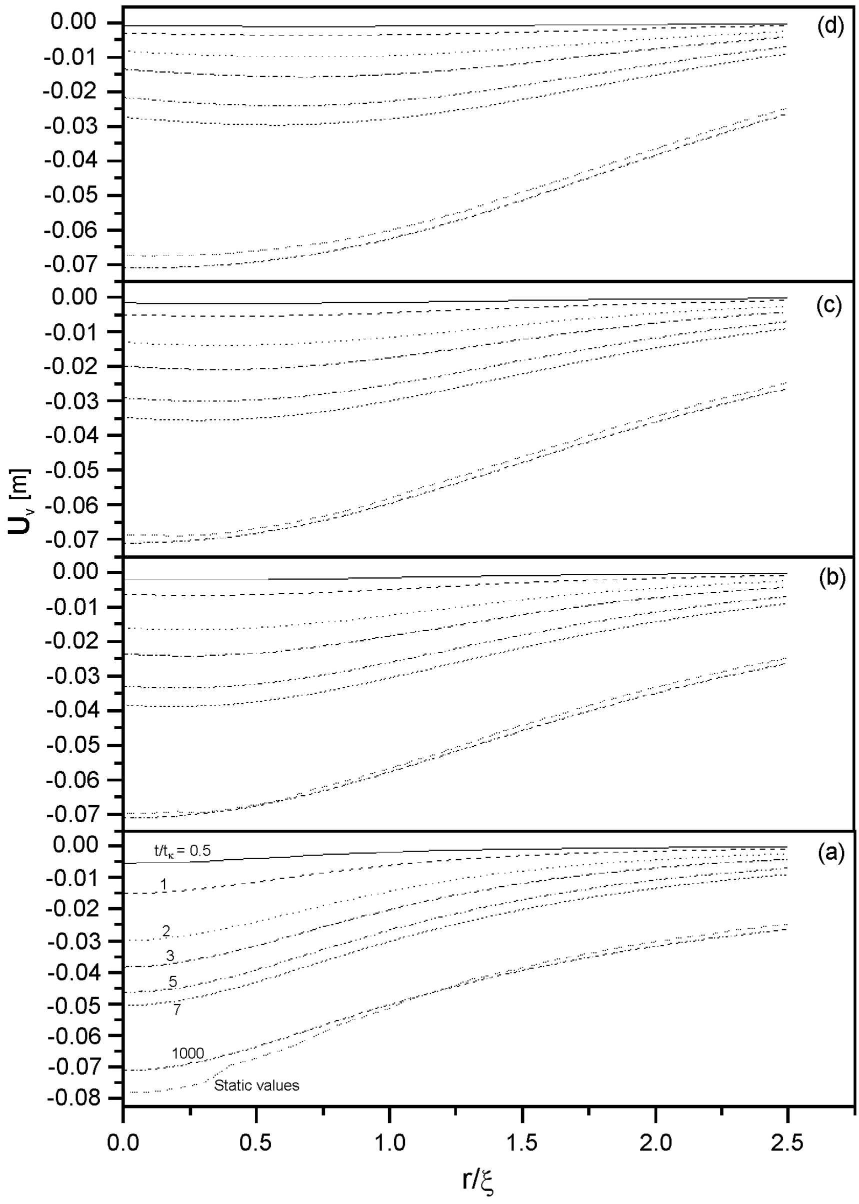

Fig. 2. Thermo-viscoelastic vertical displacement in meters computed for different time values and considering (a) a flat surface, and (b)-(d) axis-symmetric volcanic cone with an average slope of the flanks of 15°, 20° and 30°, respectively. t_κ is the decay time defined in the text.

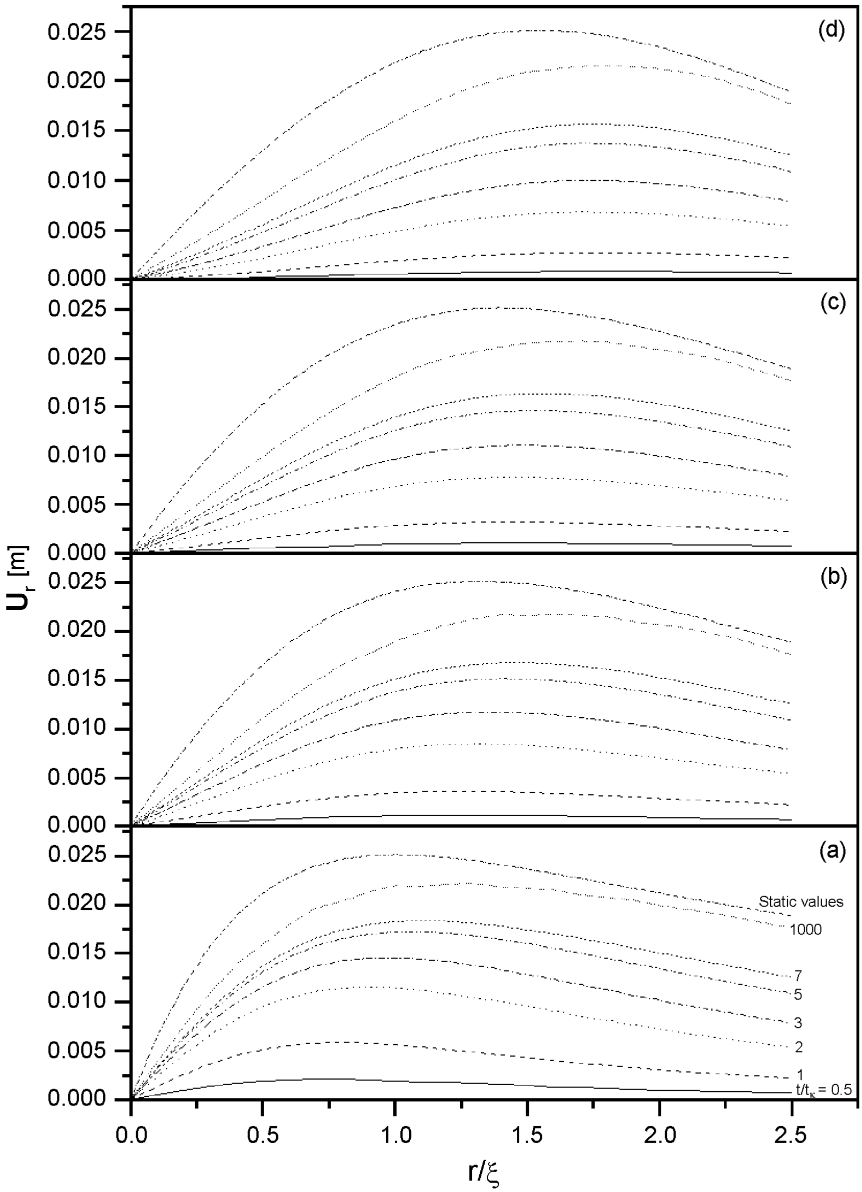

Fig. 3. Thermo-viscoelastic radial displacement in meters computed for different time values and considering (a) a flat surface, and (b)-(d) axis-symmetric volcanic cone with an average slope of the flanks of 15°, 20° and 30°, respectively. t_κ is the decay time defined in the text.

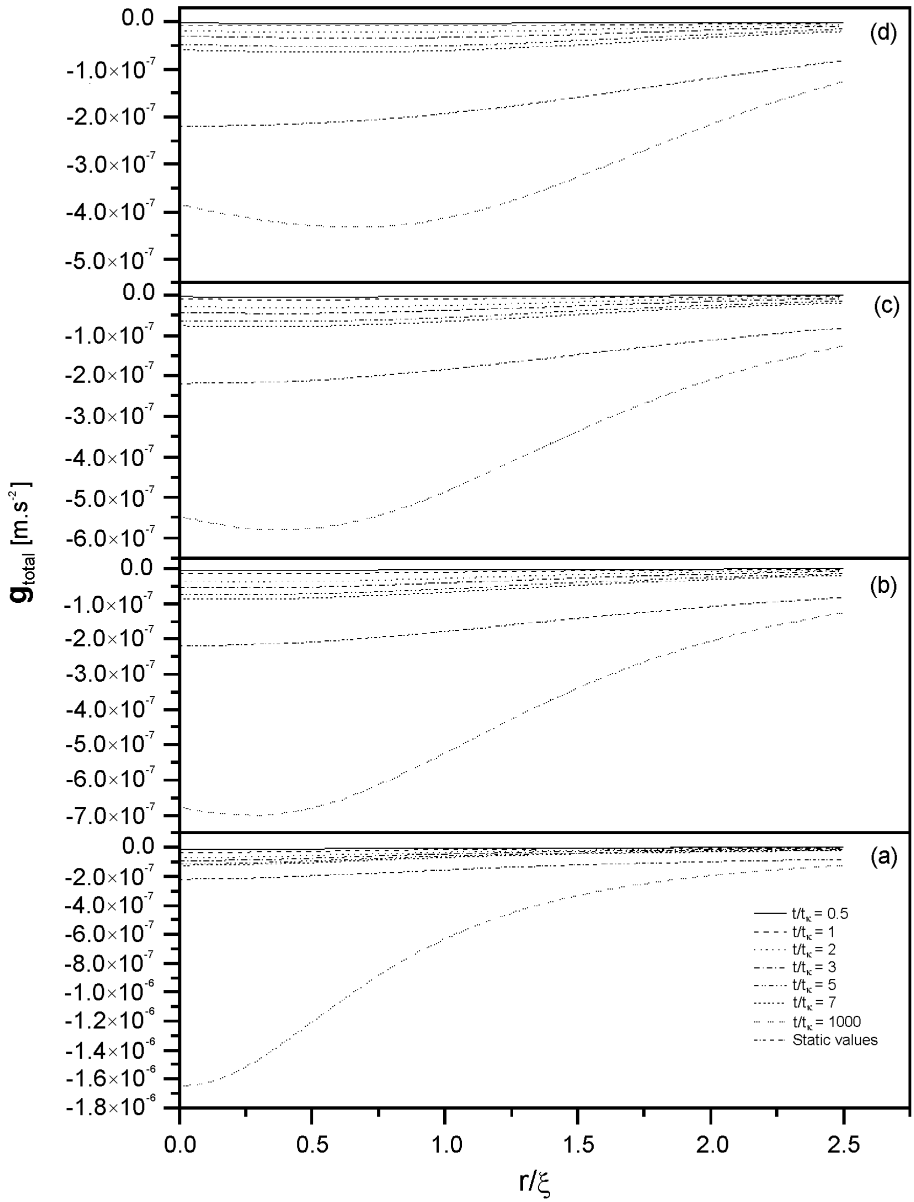

Fig. 4. Thermo-viscoelastic gravity changes in ms^{-2} computed for different time values and considering (a) a flat surface, and (b)-(d) axis-symmetric volcanic cone with an average slope of the flanks of 15°, 20° and 30°, respectively. t_κ is the decay time defined in the text.

of an elastic halfspace due to a point source at depth and assume that topography does not significantly affect the results. Considering previous results obtained by other authors for elastic (*Williams and Wadge, 1998, 2000; Cayol and Cornet, 1998*) and viscoelastic media (*Folch et al., 2000*) we have included topographic effects in the thermo-viscoelastic model. We have used an approximate methodology. This methodology permits we still have an analytical solution that allows to solve the inverse problem. With the methodology described above we can observe the reduction of vertical displacements in regions with higher topography due to the greater distance from the source of heat to the free surface. In volcanic areas of greater relief the perturbation of the thermo-viscoelastic solution (deformation and total gravity anomaly) due to topography can be quite significant. Therefore we have demonstrated that the topography may significantly affect the surface displacements and gravity changes computed for a magma chamber represented by a heat point source. Thus we can conclude that any model that neglects the topographic effect could cause a significant error in the estimation of surface displacements and gravity changes, or in the determination of the characteristics of the intrusion if we use the model to solve the inverse problem.

Acknowledgments. The author is grateful to VEGA, the Slovak Grant agency (project No. 2/3057/23), and to the Science and Technology Assistance Agency of the Slovak Republic (contract No. APVT-51-002804) for the partial support of this work.

References

Bonafede M., Dragoni M., Querini F., 1986: Displacements and stress fields produced by a centre of dilatation and by a pressure source in a viscoelastic half space: application to the study of ground deformation and seismic activity at campi Flegrei, Italy. Geophys. J. R. Astron. Soc., **87**, 455–485.

Brimich L., 1998: Study of the slow deformations of the earth's crust observed at the Vyhne tidal station. Contr. Geophys. Geod., **28**, 3, 147–160.

Brimich L., 2000: Thermoviscoelastic models of the deformations and gravity changes due to anomalous source of heat. Acta geod. et geoph. Hung., **35**, 37–48.

Carlslaw H. S., Jaeger J. C., 1959: Conduction of heat in solids. Oxford, Clarendon Press.

Cayol V., Cornet F. H., 1998: Effects of topography on the interpretation of the deformation field of prominents volcanoes: Application to Etna. Geophys. Res. Lett., **25**, 1979–1982.

Fernández J., Rundle J. B., 1994: Gravity changes and deformation due to a magmatic intrusion in a two-layered crustal model, J. Geophys. Res., **99**, 2737–2746.

Fernández J., Rundle J. B., Granell R. d. R., Yu T. T., 1997: Programs to compute deformation due to a magma intrusion in elastic-gravitational layered Earth models, Comput. Geosci., **23**, 231–249.

Fernández J., Tiampo K. F., Rundle J. B., 2001: Viscoelastic displacement and gravity changes due to point magmatic intrusions in a gravitational layered solid Earth. Geophys. J. Int., (in press).

Folch A., Fernández J., Rundle J. B., Marti J., 2000: Ground deformation in a viscoelastic medium composed of a layer overlying a half-space: a comparison between point and extended sources. Geophys. J. Int., **140**, 37–50.

Hvoždara M., Brimich L., 1991: Thermoelastic deformation field due to magmatic bodies. Contr. Geophys. Inst. Slov. Acad. Sci., **21**, 59–79.

Hvoždara M., 1992: Thermo-viscoelastic deformation field due to a point source of heat in the halfspace. Contr. Geophys. Inst. Slov. Acad. Sci., **22**, 48–66.

Hvoždara M., Rosa K., 1979: Geodynamic effects of thermo-elastic stresses due to a linear heat source. In: V. Babuška and J. Plančár (Eds), Geodynamic Investigation in Czechoslovakia, Bratislava, Veda, 53–63.

Hvoždara M., Rosa K., 1980: Stresses and displacements due to a stationary point source of heat in an elastic half-space. Studia Geophys. Geod., **24**, 51–59.

Nowacki W., 1962: Thermoelasticity. Oxford, Pergamon Press.

Michalewizc Z., 1994: Genetic Algorithms + Data structures = Evolution Programs. Springer, Berlin, Heidelberg, New York.

Rundle J. B., 1982: Deformation, gravity and potential changes due to volcanic loading of the crust. J. Geophys. R., **87**, 10729–10744.

Rundle J. B., 1983: Correction to "Deformation, gravity, and potential changes due to volcanic loading of the crust", J. Geophys. Res., **88**, 10647–10652.

Tiampo K., Rundle J. B., Fernández J., Langbein P., 2000: Spherical and ellipsoidal volcanic sources at Long Valley Caldera, California, using a Genetic Algorithm inversion tecnique. J. Volcanol. Geotherm Res., **102**, 189–206.

Williams C. A., Wadge G., 1998: The effects of the topography on magma chamber deformation models: Application to Mt. Etna and radar interferometry, Geophys. Res. Lett., **25**, 1549–1552.

Williams C. A., Wadge G., 2000: An accurate and efficient method for including the effects of topography in three-dimensional elastic models of ground deformation with applications to radar interferometry. J. Geophys. Res., **105**, B4, 8103–8120.

Yu T. T., 1995: Crustal deformation due to dipping fault in elastic gravitational layer overlying viscoelastic gravitational half space: Models and applications, Ph. D. Thesis, Univ. of Colo., Boulder.

Yu T. T., Fernández J., Rundle J. B., 1998: Inverted the parameters of an earthquake-ruptured fault with a genetic algorithm. Compu. Geosci., **24**, 173–182.