Geothermal anomaly due to a cylindrical obstacle buried in the halfspace with groundwater flow

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A b s t r a c t : We present analytical solution of two coupled potential problems: groundwater flow and geothermal disturbance affected by an partly permeable circular cylinder buried in the uniform halfspace. The solution is performed in the bipolar coordinate system and physical fields are represented by Fourier series. Numerical results show disturbance of the velocity of groundwater flow and combined refraction and convective anomalies in geothermal gradient around the cylinder.

Key word: geothermics, groundwater flow, hydrothermal anomalies, analytical methods

1. Introduction

For geohydrothermal reservoir exploitation modelling of geothermal field disturbances due to groundwater flow around various obstacles below the surface is important. The cylindrical obstacle of radius a, buried at the depth h is one model which can be treated analytically, using method of Fourier separation in bipolar coordinate system. The situation is depicted in Fig. 1 as planar flow in plane x, z. The unperturbed velocity field far away from the cylinder is supposed to be uniform, in x-direction: $\boldsymbol{V}_0 \equiv (V_0, 0, 0)$, so its potential is:

$$U_0(x, z) = -V_0 \cdot x. \tag{1}$$

The unperturbed temperature field is assumed corresponding to a uniform heat flow in z-direction, so its dependence is linear:

$$T_0(x, z) = q_0 \cdot z/\lambda_1 \tag{2}$$

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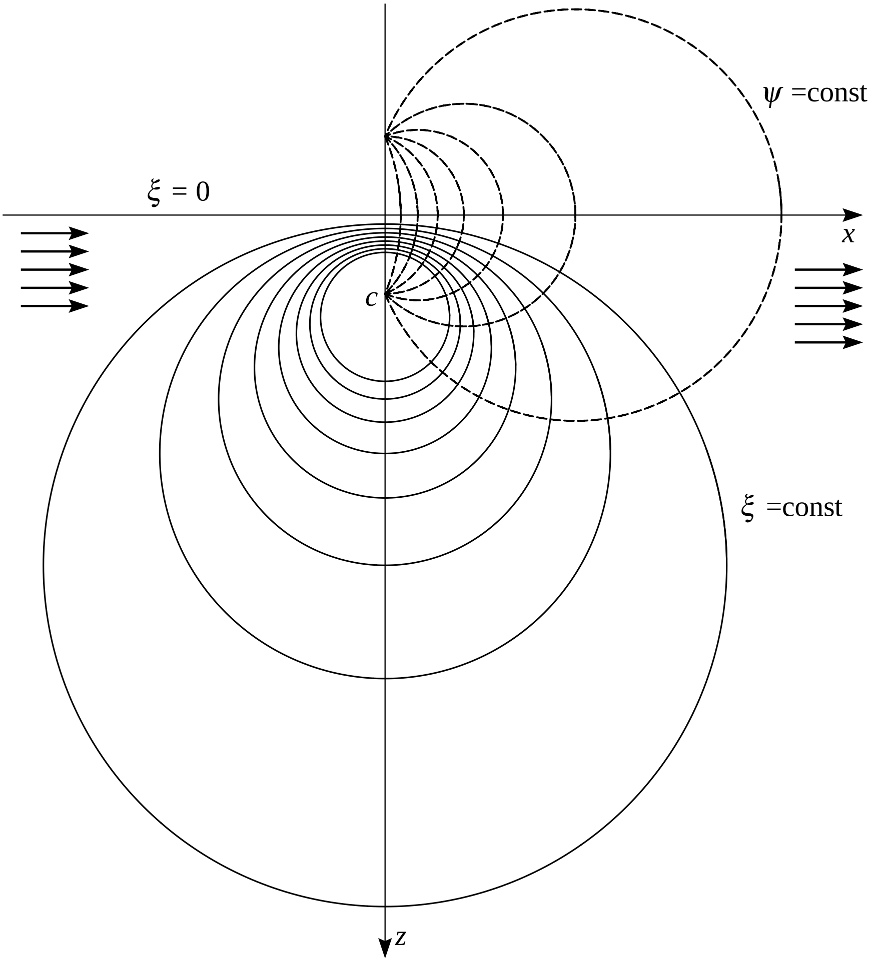

Fig. 1. The bipolar coordinate system (ξ, ψ) with lines ξ = const (full) and ψ = const (dashed), respectively.

where $q_0 = \lambda_1(\partial T_0/\partial z)$ is heat flow density and λ_1 being coefficient of heat conductivity in halfspace. The axis y is parallel to the axis of circular cylinder, so we have planar potential problem for groundwater potential $U(x, z)$, as well as for temperature field $T(x, z)$. It is clear, that the disturbance of the groundwater velocity field will produce a disturbance in heat flow, and the material contrast of the cylinder will produce another one, the so-called refraction anomaly of the normal temperature field (2).

2. Potential of the velocity field

As was noted above, we shall employ the bipolar orthogonal coordinate system (BCS) (ξ, ψ, y) related to the Cartesian coordinates by the formulae of *Arfken (1966)*:

$$x = \frac{c \sin\psi}{\operatorname{ch}\xi - \cos\psi}, \quad z = \frac{c \operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\psi}, \quad y = y, \tag{3}$$

$c = \sqrt{h^2 - a^2}$ is depth of the pole line ($\xi = +\infty$) of the BCS. The Laplace equation for general potential $U(\xi, \psi)$ is:

$$\nabla^2 U(\xi, \psi) = \frac{1}{c^2} (\operatorname{ch}\xi - \cos\psi)^2 \left[\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \psi^2} \right] = 0,$$

which reduces into simple equation:

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \psi^2} = 0 \; . \tag{4}$$

Particular solution with 2π periodicity in coordinate ψ is:

$$U_n(\xi, \eta) = \left\{ \begin{matrix} e^{-n\xi} \\ e^{+n\xi} \end{matrix} \right\} \left\{ \begin{matrix} \cos n\psi \\ \sin n\psi \end{matrix} \right. \; . \tag{5}$$

For the uniqueness of further explanation we will link our BCS to the geometrical parameters of the model. From Eqs (3) it is clear that the surface plane $z = 0$ corresponds to the value $\xi = 0$, while coordinate ψ there varies along the x axis according to formula:

$$\frac{c \sin\psi}{1 - \cos\psi} = x, \quad \text{i.e. } \operatorname{tg}(\psi/2) = c/x \, . \tag{6}$$

Now we determine value ξ in order to match the coordinate surface $\xi = \xi_1$ with the circle having centre at the point $x = 0$, $z = h$:

$$(z - h)^2 + x^2 = a^2 \, . \tag{7}$$

Using the transformation relations (3) we can find, that

$$\frac{z}{x} = \frac{\operatorname{sh}\xi}{\sin\psi}, \quad \sin\psi = \frac{x}{z} \operatorname{sh}\xi \, .$$

By elimination of ψ from Eqs (3) we can find general relation between (x, z) and ξ:

$$(z - c\coth\xi)^2 + x^2 = c^2/\operatorname{sh}^2\xi\,, \tag{8}$$

which means that curves $\xi = \text{const} \equiv k_0$ are circles of radius equal to $c/\operatorname{sh} k_0$ and centre in the depth $h = c\cdot\coth k_0$. This means in comparison with (7) that we have for the surface of the cylinder, coordinate ξ_1 which is linked to h and a as follows:

$$h = c\cdot\coth\xi_1, \quad a = c/\operatorname{sh}\xi_1\,. \tag{9}$$

From the quadratic values of h and a we obtain:

$$h^2 - a^2 = \frac{c^2\operatorname{ch}^2\xi_1}{\operatorname{sh}^2\xi_1} - \frac{c^2}{\operatorname{sh}^2\xi_1} = \frac{c^2\operatorname{sh}^2\xi_1}{\operatorname{sh}^2\xi_1} = c^2\,,$$

i.e. the parameter c of the biaxial coordinate system:

$$c^2 = h^2 - a^2, \quad c = \sqrt{h^2 - a^2}\,, \tag{10}$$

which means that the pole of BCS is closer to the surface than the axis of the cylinder. Using relation $a\cdot\operatorname{sh}\xi_1 = c$ we can find that value ξ_1 is linked to h, a, c as

$$e^{\xi_1} = (c+h)/a\,, \quad \xi_1 = \ln[(c+h)/a]\,. \tag{11}$$

Here we can see, that if $a \to 0$, then $\xi_1 \to +\infty$ and $c = h$, so the cylinder is degenerated to the straight line in the depth h and parallel to the y-axis.

Now we present expansion of the unperturbed potential (1) into series with particular solution (5). We have to find the Fourier expansion of $U_0(x) = -V_0\cdot x$, which in BCS means:

$$U_0(\xi,\eta) = -V_0\frac{c\sin\psi}{\operatorname{ch}\xi - \cos\psi}\,. \tag{12}$$

We turn our attention to the function

$$\frac{\sin\psi}{\operatorname{ch}\xi - \cos\psi} = \frac{2\sin\psi}{e^{\xi} + e^{-\xi} - 2\cos\psi} = \frac{2e^{-\xi}\sin\psi}{1 + e^{-2\xi} - 2e^{-\xi}\cos\psi}\,.$$

Now we use the formula 1.447.1 from *(Gradstejn and Ryzhik, 1971)* which reads:

$$\frac{2p\sin\psi}{1-2p\cos\psi+p^2} = 2\sum_{n=1}^{\infty} p^n \sin n\psi\,, \quad |p|<1\,.$$

We put $p = e^{-\xi}$ and we have an expansion for (12):

$$U_0(\xi,\psi) = -2V_0 c \sum_{n=1}^{\infty} e^{-n\xi}\sin n\psi\,. \tag{13}$$

Due to the presence of the cylinder this potential will be changed to function $U_1(\xi,\psi)$ for the exterior of the cylinder $\xi \in \langle 0,\xi_1\rangle$ and to function $U_2(\xi,\psi)$ in the interior of the cylinder $\xi \in \langle \xi_1, +\infty\rangle$.

On the free surface $\xi = 0$ the boundary condition must be satisfied. So:

$$[\partial U_1(\xi,\psi)/\partial\xi]_{\xi=0} = 0\,, \tag{14}$$

this condition is satisfied by $U_0(\xi,\psi)$ as follows from (12), so we must add to $U_0(\xi,\psi)$ perturbing potential $U_1^*(\xi,\psi)$ with dependence on ξ as $\mathrm{ch}(n\xi)$, so we have:

$$U_1(\xi,\psi) = U_0(\xi,\psi) - 2V_0 c\sum_{n=1}^{\infty} A_n\,\mathrm{ch}(n\xi)\sin n\psi\,. \tag{15}$$

The velocity potential in the interior of the cylinder must be bounded even for $\xi \to +\infty$, so we have:

$$U_2(\xi,\psi) = U_0(\xi,\psi) - 2V_0 c\sum_{n=1}^{\infty} C_n e^{-n\xi}\sin n\psi\,. \tag{16}$$

The terms with $\cos n\psi$ will not occur in the view of $\sin(n\psi)$ dependence of Fourier series (13). We suppose the diffusivity coefficient σ_1 in the halfspace and σ_2 in the cylinder. Then the boundary condition must be satisfied at the surface $\xi = \xi_1$:

$$\begin{aligned} &U_1(\xi,\psi)\big|_{\xi=\xi_1} = U_2(\xi,\psi)\big|_{\xi=\xi_1}\,,\\ &\left.\frac{\partial U_1(\xi,\psi)}{\partial\xi}\right|_{\xi=\xi_1} = \frac{\sigma_2}{\sigma_1}\left.\frac{\partial U_2(\xi,\psi)}{\partial\xi}\right|_{\xi=\xi_1}\,, \end{aligned} \tag{17}$$

Using Fourier series technique we obtain two linear equations for determination of A_n and C_n:

$$A_n \,\mathrm{ch}(n\xi_1) - C_n e^{-n\xi_1} = 0\,,$$
$$A_n \,\mathrm{sh}(n\xi_1) + \frac{\sigma_2}{\sigma_1} C_n e^{-n\xi_1} = \left(1 - \frac{\sigma_2}{\sigma_1}\right) e^{-n\xi_1}\,.$$

The solution is easy and gives:

$$A_n = \frac{(1-\sigma_2/\sigma_1)e^{-n\xi_1}}{\mathrm{sh}(n\xi_1) + \sigma_2/\sigma_1\,\mathrm{ch}(n\xi_1)} = \frac{2k_{12}\,e^{-2n\xi_1}}{1 - k_{12}\,e^{-2n\xi_1}}\,, \tag{18}$$

where $k_{12} = (1-\sigma_2/\sigma_1)/(1+\sigma_2/\sigma_1)$ is the coefficient of the diffusivity contrast. For coefficient C_n we have:

$$C_n = A_n \cdot e^{n\xi_1}\,\mathrm{ch}(n\xi_1) = k_{12}\frac{(1+e^{-2n\xi_1})}{1-k_{12}e^{-2n\xi_1}}\,. \tag{19}$$

Thus we have suitable formulae for perturbation potentials:

$$U_1^*(\xi,\psi) = -4CV_0k_{12}\sum_{n=1}^{\infty}\frac{e^{-2n\xi_1}\cosh(n\xi)}{1-k_{12}e^{-2n\xi_1}}\sin n\psi\,, \tag{20}$$

$$U_2^*(\xi,\psi) = -2CV_0k_{12}\sum_{n=1}^{\infty}\frac{(1+e^{-2n\xi_1})}{1-k_{12}e^{-2n\xi_1}}e^{-n\xi}\sin n\psi\,. \tag{21}$$

It is obvious, that for $k_{12} = 0$, i.e. $\sigma_2 = \sigma_1$ these perturbation potentials are zero, since the cylinder has the same diffusivity as the surrounding halfspace.

3. Calculation of velocity field and convective disturbance of heat flow

For the potentials calculated in the previous section we need derivatives with respect to x and z in order to determine the velocity field in Cartesian coordinates:

$$V_x = -\partial U/\partial x\,, \qquad V_z = -\partial U/\partial z\,. \tag{22}$$

Taking the derivatives of the primary potential U_0 is easy. Since both potentials $U^*_{12}(\xi,\psi)$ are expressed in BCS (ξ and ψ), we must calculate more complicated derivatives namely:

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \psi}\frac{\partial \psi}{\partial x}\,, \qquad \frac{\partial U}{\partial z} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial U}{\partial \psi}\frac{\partial \psi}{\partial z}\,. \tag{23}$$

Since derivatives with respect to ξ,ψ of series (20), (21) are easy, we must give derivatives of (ξ,ψ) with respect to (x,z). We adopted formulae of *Arfken (1966)* according to which

$$\xi = \frac{1}{2}\ln\frac{(z+c)^2+x^2}{(z-c)^2+x^2}\,, \quad \psi = \mathrm{arctg}\,\frac{2xc}{(z^2-c^2)+x^2}\,. \tag{24}$$

After some calculus we obtain:

$$\frac{\partial \xi}{\partial x} = \frac{x}{\rho_2^2} - \frac{x}{\rho_1^2}\,, \quad \frac{\partial \xi}{\partial z} = \frac{z+c}{\rho_2^2} - \frac{z-c}{\rho_1^2}\,, \tag{25}$$

where $\rho_1^2 = (z-c)^2+x^2$, $\rho_2^2 = (z+c)^2+x^2$. Similar derivatives of ψ are:

$$\frac{\partial \psi}{\partial x} = \frac{2c[(z^2-c^2)-x^2]}{4x^2c^2+[(z^2-c^2)+x^2]^2}\,, \quad \frac{\partial \psi}{\partial z} = \frac{-4cxz}{4x^2c^2+[(z^2-c^2)+x^2]^2}\,. \tag{26}$$

For completeness we give partial derivatives of $U_1^*(\xi,\psi)$ and $U_2^*(\xi,\psi)$:

$$\begin{aligned}
\frac{\partial U_1^*}{\partial \xi} &= -4cV_0k_{12}\sum_{n=1}^{\infty}\frac{ne^{-2n\xi_1}\,\mathrm{sh}(n\xi)}{1-k_{12}\,e^{-2n\xi_1}}\sin(n\psi)\,,\\
\frac{\partial U_1^*}{\partial \psi} &= -4cV_0k_{12}\sum_{n=1}^{\infty}\frac{ne^{-2n\xi_1}\,\mathrm{ch}(n\xi)}{1-k_{12}\,e^{-2n\xi_1}}\cos(n\psi)\,.
\end{aligned} \tag{27}$$

For the interior of cylinder we have:

$$\begin{aligned}
\frac{\partial U_2^*}{\partial \xi} &= +2cV_0k_{12}\sum_{n=1}^{\infty}\frac{(1+e^{-2n\xi_1})n}{1-k_{12}\,e^{-2n\xi_1}}e^{-n\xi}\sin(n\psi)\,,\\
\frac{\partial U_2^*}{\partial \psi} &= -2cV_0k_{12}\sum_{n=1}^{\infty}\frac{(1+e^{-2n\xi_1})}{1-k_{12}\,e^{-2n\xi_1}}e^{-n\xi}n\cos(n\psi)\,.
\end{aligned} \tag{28}$$

According to *Carslaw and Jaeger (1959)* the convective part of the disturbance in the vertical component of the heat flow is:

$$q_z^* = -f_z^* = -\rho C_v T(x,z) \cdot V_z = +\rho C_v T(x,z) \frac{\partial U}{\partial z}\,, \tag{29}$$

where ρ is density and C_v is specific heat of the fluid. The negative sign is according to the convection adopted in geothermics

$$q_z = +\lambda(\partial T / \partial z)\,. \tag{30}$$

4. The effect contrast of heat flow conductivity (refraction effect)

It is clear, that the heat flow will be affected by the different heat flow conductivity of the cylinder λ_2 and the surrounding halfspace medium λ_1. This is known as the refraction effect of the heat flow. The undisturbed temperature far from the cylinder was considered in the form (2), which according to transformation formulae (3) gives:

$$T_0(\xi,\psi) = g_0 \frac{\operatorname{sh}\xi}{\operatorname{ch}\xi - \cos\psi}\,, \tag{31}$$

where $g_0 = q_0 \cdot c/\lambda_1$ is undisturbed temperature at the depth $z = c$.

The steady temperature field in the region without heat sources obeys the Laplace equation, so we can use particular solution, i.e. functions (5) to express temperature outside the cylinder $T_1(\xi,\psi)$ and inside it $T_2(\xi,\psi)$:

$$T_1(\xi,\psi) = T_0(\xi,\psi) + g_0 \sum_{n=1}^{\infty} B_n \operatorname{sh}(n\xi) \cos n\psi\,, \tag{32}$$

$$T_2(\xi,\psi) = T_0(\xi,\psi) + g_0 \sum_{n=1}^{\infty} G_n e^{-n\xi} \cos n\psi\,. \tag{33}$$

The temperature $T_1(\xi,\psi)$ attains zero value on the planar boundary $\xi = 0$ and on the surface of the cylinder $\xi = \xi_1$ there must be continuous temperature and normal component of the heat flow, i.e.:

$$T_1(\xi_1,\psi) = T_2(\xi_1,\psi)\,, \tag{34}$$

$$[\partial T_1/\partial\xi]_{\xi_1} = (\lambda_2/\lambda_1)\,[\partial T_2/\partial\xi]_{\xi_1}\,. \tag{35}$$

Now we need expansion of $T_0(\xi,\psi)$ into Fourier series of particular functions (5) which we obtain easily as follows:

$$T_0(\xi,\psi) = 2g_0 \frac{\operatorname{sh}\xi}{e^{\xi}+e^{-\xi}-2\cos\psi} = \\ = g_0 \frac{1-e^{-2\xi}}{1+e^{-2\xi}-2e^{-\xi}\cos\psi} = g_0\left[1+2\sum_{n=1}^{\infty} e^{-n\xi}\cos n\psi\right]. \tag{36}$$

Here we have employed modification of the formula 1.447.3 from *Gradstejn and Ryzhik (1971).*

Using Fourier series theory we can realize from boundary conditions (34) and (35) that we have a system of two linear equations:

$$B_n \operatorname{sh}(n\xi_1) = G_n e^{-n\xi_1}, \\ -2e^{-n\xi_1} + B_n \operatorname{ch}(n\xi_1) = -(\lambda_2/\lambda_1)[G_n e^{-n\xi_1} + 2e^{-n\xi_1}]. \tag{37}$$

After simple algebra we have:

$$B_n = \frac{4\gamma_{12}\, e^{-2n\xi_1}}{1+\gamma_{12}\, e^{-2n\xi_1}}, \tag{38}$$

where $\gamma_{12} = (1-\lambda_2/\lambda_1)/(1+\lambda_2/\lambda_1)$. For the interior of the cylinder we have coefficients:

$$G_n = 2\frac{\gamma_{12}(1-e^{-2n\xi_1})}{1+\gamma_{12}\, e^{-2n\xi_1}}. \tag{39}$$

Now we have the possibility to calculate temperature and heat flow density corresponding to $T_1(\xi,\psi)$ or $T_2(\xi,\psi)$. With respect to the convective part of the vertical component of the heat flow, we obtain the net vertical heat flow density inside or outside the cylinder

$$\overline{q}_z = \lambda\frac{\partial T}{\partial z} + \rho\, C_v\, T(x,z)\frac{\partial U}{\partial z}, \tag{40}$$

where z derivative must be calculated similarly to formula (23).

5. Numerical calculations

For our numerical calculations using the derived formulae we adopted the following parameters of the model:

- radius of the cylinder $a = 1$ (km)
- depth of its axis $h = 1.5 \times a$
- ratio of the diffusivity coefficients $\sigma_2/\sigma_1 = 0.05$.

This low value of σ_2/σ_1 corresponds to the case when the cylinder is almost impermeable for the ground water flow. For the coefficients of heat conductivity we considered two ratios of $\lambda_2/\lambda_1 = 3$ and 0.3, respectively, which means good and low thermal conductivity of the cylinder. The product ρC_v (density multiplied by the specific heat of the fluid) is for the water equal to 4.19×10^6 $\mathrm{J\,m^{-3}\,K^{-1}}$. Since the practical velocities V_{oz} of the ground water flow at the depths about 1–2 km are few meters per year, we have chosen value $\rho C_v \cdot V_{oz} = 0.80\,\mathrm{W\,m^{-2}\,K^{-1}}$, which corresponds to the velocity $V_{oz} \approx 6\,\mathrm{m/y} = 1.9013 \cdot 10^{-7}\,\mathrm{m\,s^{-1}}$. Then the quantity $q_z^* = \rho C_v (V_{oz}/V_0)\, T(x,z) \partial U/\partial z$ will reflect ratio of the convective heat transfer to the "normal" $q_0 = 1$ (provided the velocity potentials U_1, U_2 are calculated for normal horizontal velocity $V_0 = 1\,\mathrm{m/s}$. Let us stress, that the main purpose of our paper is the study of general features of the interaction between velocity and thermal fields.

The results for the cylinder, which is to groundwater impermeable, but with high thermal conductivity ($\lambda_2/\lambda_1 = 3$), are presented in the Fig. 2. The top isoline map presents equipotential lines of the velocity potentials $U_1(x,z)\; U_2(x,z)$ according to formulae from Section 2. The thin lines in this map depict the velocity directions (force lines) of $\boldsymbol{V}_{1,2} = -\operatorname{grad} U_{1,2}(x,z)$ which were calculated by the formulae from Section 3. We can see that the lines of groundwater flow "avoid to enter" into cylinder, since we put the ratio $\sigma_2/\sigma_1 = 0.05$. The potential pattern is antisymmetric with respect to the plane $x = 0$. The velocity components V_x are symmetric, while V_z are antisymmetric to this plane.

The middle map in Fig. 2 presents isotherms in our model, together with isolines official $\overline{q}_z/q_0$, where $\overline{q}_z$ represents superposition of conductive and convective parts of the heat flow. We can see that isotherms are sightly bowed to the surface, since $\lambda_2/\lambda_1 = 3$ and isolines of $\overline{q}_z/q_0$ have quite complicated course, especially near the surface of the cylinder. The region of $\overline{q}_z/q_0 > 1.0$ appears above the cylinder, as well as on the right side below it. The pattern of isolines not symmetric with respect to the $x = 0$ plane is the effect of the convection. In the bottom graphs in Fig. 2 there are

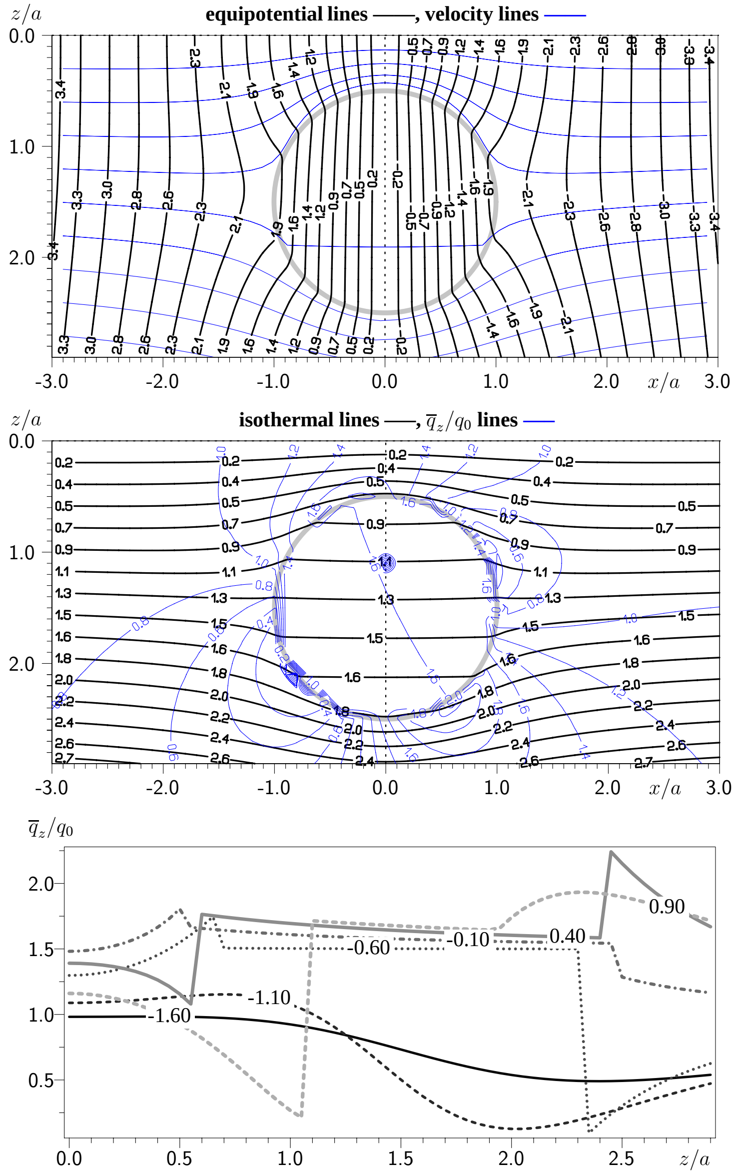

Fig. 2. Results of numerical calculation for the impermeable cylinder: $h/a = 1.5, \sigma_2/\sigma_1 = 0.05$ and good thermal conductivity, $\lambda_2/\lambda_1 = 3$. The top figure presents equipotential lines (full) and velocity lines (gray) of the groundwater flow. The middle map presents isotherms (full) and $\overline{q}_z/q_0$ lines (gray). The bottom graphs present more detailed vertical profiles $\overline{q}_z/q_0$ along "boreholes" at $x/a = -1.6, -1.1, -0.6, -0.1, 0.4, 0.9$.

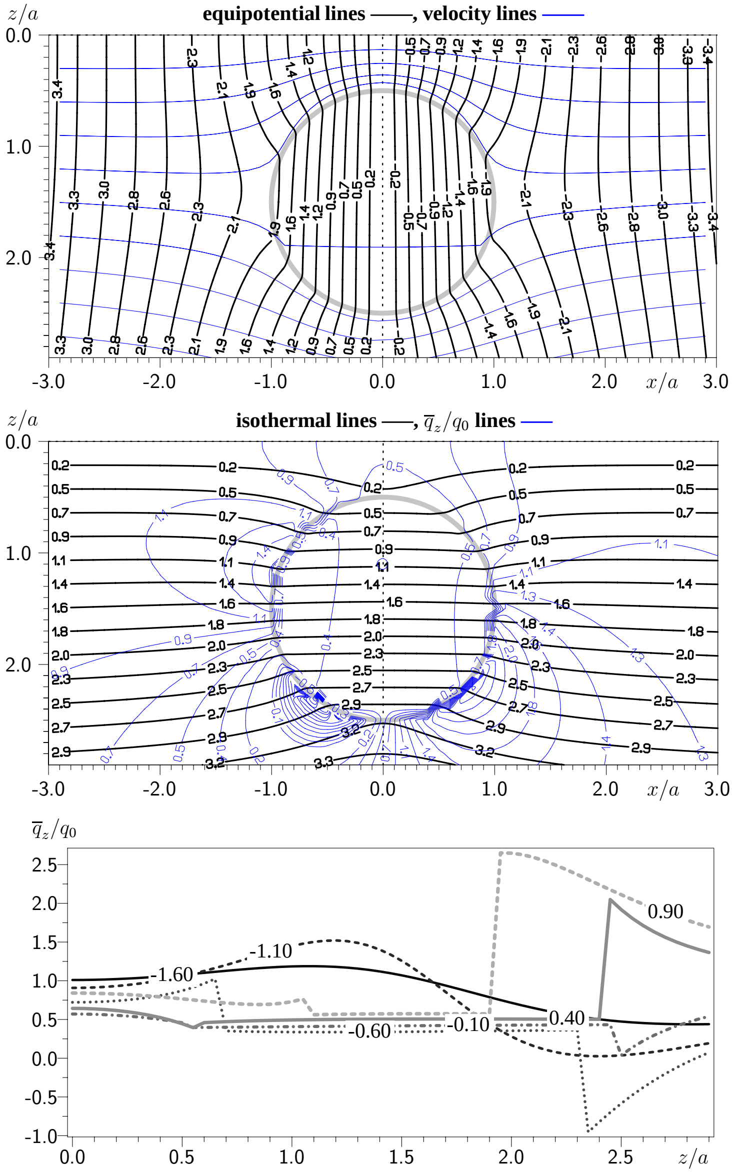

Fig. 3. The same as in Fig. 2, but for low conductive cylinder $\lambda_2/\lambda_1 = 0.3$.

plotted vertical courses of $\overline{q}_z/q_0$ for six "boreholes" situated at $x/a = -1.6$, -1.1, -0.6, -0.1, 0.4, 0.9. We can see, that most effective boreholes for hydrothermal exploitation are e.g. for $x/a = -0.6$, -0.1, 0.4 which can be effective even being shallow $z \in \langle 0, h-a)$.

In the Fig. 3 there are plotted similar graphs but for the low conductive cylinder for the heat transfer, i.e. $\lambda_2/\lambda_1 = 0.3$. The filtration ratio σ_2/σ_1 is again equal to 0.05, so the velocity field is the same as in Fig. 2. But temperature field is substantially different. We can see, that the region with $\overline{q}_z/q_0 > 1$ is much smaller in comparison with Fig. 2, it resides outside the cylinder at depths $z/a \sim 0.7 - 1.6$ and $x/a \lesssim -1$. Another region with $\overline{q}_z/q_0$ is in the right part of figure, but at depths $z > h$.

We can conclude, that better situation for hydrothermal resources is provided by structures with $\sigma_2/\sigma_1 \ll 1$ which push groundwater flow toward the surface mainly above the cylinder and with $\lambda_2/\lambda_1 \gtrsim 2$ which enable flows of the conductive heat flow also above the cylinder. This situation is illustrated in Fig. 2.

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