Out-of-plane principal stress in plane strain/stress failure investigations – an overview of its relevance for various failure criteria

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Abstract: The paper exposes the role of the out-of-plane principal stress in the investigations of (brittle) failure of isotropic linear elastic continuum using plane strain (or plane stress) approximations. The plots in (σ_m, τ_m), (σ_3, σ_1) spaces for each of the examined criteria: the two-dimensional: Coulomb, Tresca, Hoek-Brown, and the three-dimensional ones: Drucker-Prager and linear Mogi, give an insight into the interplay of Poisson ratio and the material parameters involved in the respective criterion formula.

Key words: plane strain, plane stress, out-of-plane principal stress, Mohr circle, brittle failure criteria

1. Introduction

Although sometimes, due to some special geometric properties of the three-dimensional elastic continuum, the problem to be solved actually degenerates to two-dimensional (or plane problem), the continuum itself, nevertheless, does not cease to be fully three-dimensional and shall be dealt with accordingly. In three dimensions, the state of stress in a point can be fully described by the triplet of principal stress values and the corresponding orthogonal triplet of principal stress directions – normals of planes, on which zero shear stresses and normal stresses equal to the respective principal stresses act.

In plane problems, two of the principal directions (let us denote the corresponding principal stresses σ_1 and σ_3, $\sigma_1 \geq \sigma_3$, compression being positive)

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are parallel to the studied planar cross – section, the third one is perpendicular to it. In plane stress, the principal stress value σ_2 corresponding to that out-of-plane principal direction (out-of-plane principal stress, for brevity) is

$$\sigma_2 = 0. \tag{1}$$

In plane strain problems applied to the linearly elastic isotropic continuum, the out-of-plane principal stress value σ_2 linearly depends on the other two:

$$\sigma_2 = \nu\left(\sigma_1 + \sigma_3\right), \tag{2}$$

where ν is the Poisson ratio of the material.

Given the state of stress in a point, the material can either sustain it or fail, i.e., undergo an irreversible change of its structure.

2. Failure criteria in general

Any reasonable failure criterion can be expressed in terms of physical quantities, whose values do not change under transformation of coordinates: either in terms of invariants of stress tensor, or in terms of principal stress values.

In the formulae of three-dimensional criteria, all the three principal stress values appear at once. In the two-dimensional criteria, only couples (however, all the three couples, couple by couple) of principal stress values are involved.

The most common graphical presentation of the two-dimensional failure criteria is the one based on the use of Mohr circles. A Mohr circle (Fig. 1) illustrates the quantitative relationships between a couple of principal stresses σ_1, σ_3, whose corresponding principal directions define a plane, and the normal and tangential stresses σ, τ acting on a plane perpendicular to the former plane, whereby θ is the angle between the normal to the latter plane and the principal direction 1 *(Jaeger and Cook, 1979, p. 15)*.

Common for most shear failure criteria is the assertion that the failure occurs on a plane perpendicular to the plane containing the two principal directions corresponding to the principal stresses with the biggest difference, i.e., σ_i, σ_j, $i,\ j \in \{1,\ 2,\ 3\}$, such that:

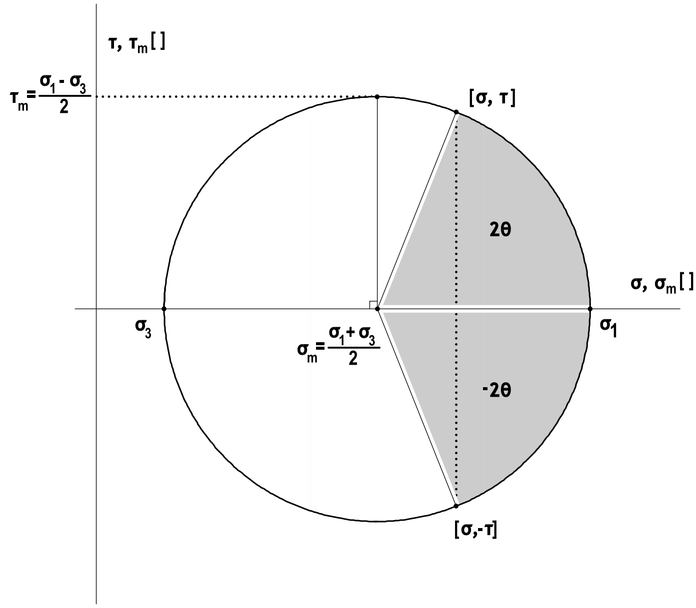

Fig. 1. The Mohr circle.

$$|\sigma_i - \sigma_j| = \max_{k,\, l \in \{1,2,3\}} |\sigma_k - \sigma_l|\,. \tag{3}$$

Let us assume the principal stresses σ_i, σ_j fulfilling (3) are σ_1, σ_3. Next, let us vary the values of σ_1, σ_3, so that at failure they produce many different Mohr circles with a common envelope (Fig. 2). The envelope curves $\tau = \pm f(\sigma)$ touch a particular circle in points $[\sigma,\ \pm\tau]$, which define two orientations of failure planes identical with, or very close to, the orientations actually observed (Fig. 3).

Thus,

$$|\tau| = f(\sigma) \tag{4}$$

is the general form of a two-dimensional failure criterion, often called also Mohr-Coulomb criterion. For $\tau \geq 0$, we can write the criterion simply as

$$\tau = f(\sigma), \tag{5}$$

and this is the form we will use in the sequel. The same criterion can be written in terms of the mean of the two in-plane principal stresses σ_m,

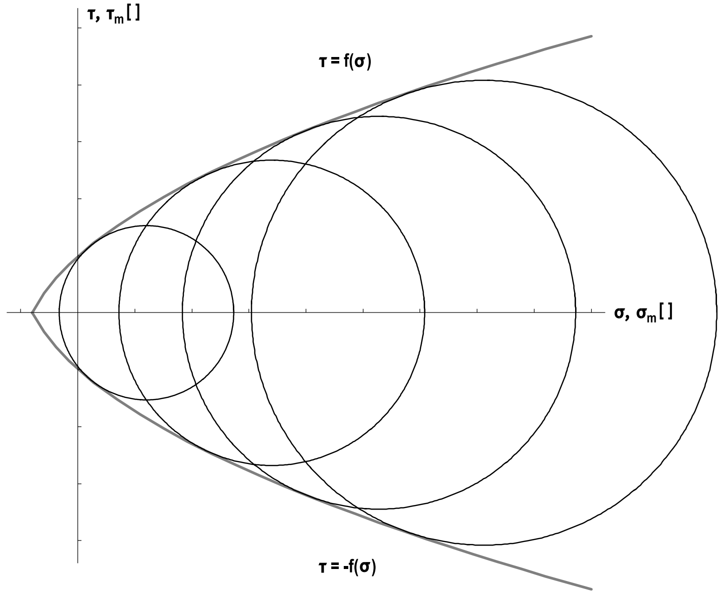

Fig. 2. The envelope of Mohr circles.

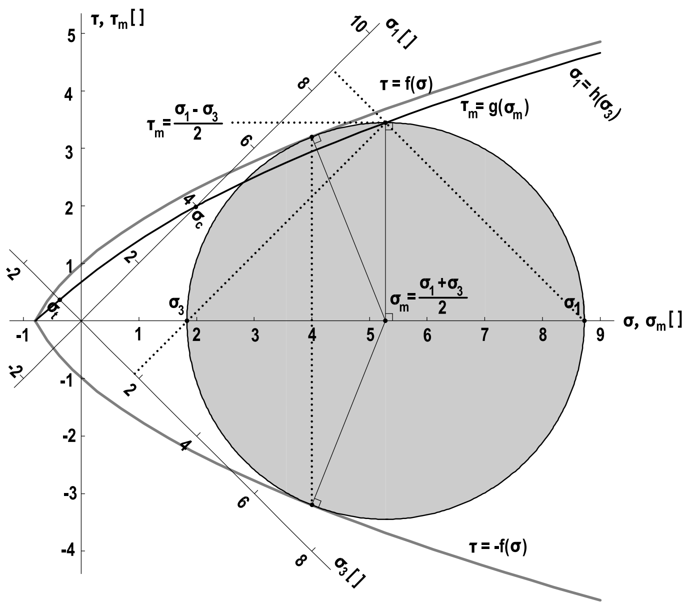

Fig. 3. The Mohr-Coulomb criterion and relationships between $f(\sigma)$, $g(\sigma_m)$ and $h(\sigma_3)$.

$\sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3)$, and the maximum in-plane shear τ_m, $\tau_m = \frac{1}{2}(\sigma_1 - \sigma_3)$, i.e., the centre and the radius of the Mohr circle, respectively, as:

$$\tau_m = g(\sigma_m), \tag{6}$$

or in terms of the principal stresses σ_1, σ_3 at failure as:

$$\sigma_1 = h(\sigma_3). \tag{7}$$

Whereas the formulae (4), (5), (6) have a clear geometric relationship to Mohr circles, the advantage of plots of (7) is the easy specification of *shear failure* mechanism region, to which the validity of all the here investigated criteria is restricted, directly by *quadrants* of the space (σ_3, σ_1), where at least one principal stress is positive (i.e., compressive).

Either of the formulae (5), (6), (7) can be obtained by fitting the experimental data in the respective space with a function of the chosen class; the remaining two formulae can then be derived, most conveniently in the form of parametric plots.

The general parametric formulae for conversions between (5) and (6) are:

$$(\sigma_m,\ \tau_m) = \left(\sigma + f(\sigma)\, f'(\sigma),\ f(\sigma)\sqrt{1 + (f'(\sigma))^2}\right), \tag{8}$$

$$(\sigma,\ \tau) = \left(\sigma_m - g(\sigma_m)\, g'(\sigma_m),\ g(\sigma_m)\sqrt{1 - (g'(\sigma_m))^2}\right). \tag{9}$$

The transformation of (6) to (7) can be written in matrix form as:

$$\begin{pmatrix} \sigma_3 \\ \sigma_1 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \sigma_m \\ \tau_m \end{pmatrix}, \tag{10}$$

which after the substitution of (6) (or (7)) yields the parametric formula for conversion of (6) to (7) (or (7) to (6), respectively).

The more complicated is the primary formulation of the criterion in its "native" space, the harder it is to eliminate the parameter from the parametric secondary reformulation in the "foreign" space. If we, for illustrative purposes, take up with the graphic form of the transformation between (6) and (7), we can take advantage of the simple geometric relationships between these spaces: (10) can be deciphered as

$$\begin{pmatrix} \sigma_3 \\ \sigma_1 \end{pmatrix} = \sqrt{2} \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \begin{pmatrix} \sigma_m \\ \tau_m \end{pmatrix}. \tag{11}$$

Thus, by anti-clockwise rotation by the angle $\frac{\pi}{4}$ and uniform stretching by the factor of $\sqrt{2}$, we transform an object in (σ_m, τ_m) space into its image in (σ_3, σ_1) space. Or, we can do the opposite: leave the object as it is, rotate the coordinate axes by $-\frac{\pi}{4}$ and rescale them by factor of $\frac{1}{\sqrt{2}}$ to obtain the coordinate axes of the (σ_3, σ_1) space (Fig. 3). In the sequel, we will use this latter approach in the graphical presentations and draw both the coordinate systems in one plot.

3. The statement of the problem

In plane problems, let us break the convention of indexing the principal stresses so that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, and introduce instead a convention of σ_1, σ_3 such that $\sigma_1 \geq \sigma_3$ for in-plane principal stresses and σ_2 for the out-of-plane principal stress.

Taking into account that σ_2 is not an independent variable, but is given as (2) for plane strain or (1) for plane stress, we have here only two independent principal stresses σ_1 and σ_3.

The two-dimensional criterion $\tau_m = g(\sigma_m)$, (where $\sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3)$, $\tau_m = \frac{1}{2}(\sigma_1 - \sigma_3)$), is only valid for $\sigma_1 \geq \sigma_2 \geq \sigma_3$, whereas the principal stresses σ_1, σ_2, σ_3 can obviously take values $\sigma_2 \geq \sigma_1 \geq \sigma_3$ or $\sigma_1 \geq \sigma_3 \geq \sigma_2$, as well. In these two latter cases, according to (3), the plane of failure is not perpendicular to the plane problem cross-section, i.e., the plane containing the principal directions 1 and 3.

Bearing this in mind, we shall, for the given plane problem, find a compound criterion covering all the values of σ_1 and σ_3:

$$\tau_m = g_C(\sigma_m) = \begin{cases} g_-(\sigma_m), & \sigma_2 \geq \sigma_1 \geq \sigma_3, \\ g(\sigma_m), & \sigma_1 \geq \sigma_2 \geq \sigma_3, \\ g_+(\sigma_m), & \sigma_1 \geq \sigma_3 \geq \sigma_2. \end{cases} \tag{12}$$

Let us now specify the subregions on which $g_C(\sigma_m)$ is defined, in terms of σ_m, τ_m:

$$\tau_m = g_C(\sigma_m) = \begin{cases} g_-(\sigma_m), & \tau_m \leq -(1-2k)\,\sigma_m, \\ g(\sigma_m), & \tau_m \geq -(1-2k)\,\sigma_m \wedge \tau_m \geq (1-2k)\,\sigma_m, \\ g_+(\sigma_m), & \tau_m \leq (1-2k)\,\sigma_m, \end{cases} \tag{13}$$

where $k = \nu$ for plane strain and $k = 0$ for plane stress.

Similarly, any three-dimensional criterion degenerates to two-dimensional, and can be expressed in the same manner.

4. The general solution

For the subregions where $\sigma_2 \geq \sigma_1 \geq \sigma_3$ or $\sigma_1 \geq \sigma_3 \geq \sigma_2$, in the case of isotropic material, the same criterion as for $\sigma_1 \geq \sigma_2 \geq \sigma_3$ is valid – however, instead of $\sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3)$ and $\tau_m = \frac{1}{2}(\sigma_1 - \sigma_3)$,

$$\sigma_{m-} = \frac{1}{2}(\sigma_2 + \sigma_3)\,, \quad \tau_{m-} = \frac{1}{2}(\sigma_2 - \sigma_3) \quad \text{for} \;\; \sigma_2 \geq \sigma_1 \geq \sigma_3, \tag{14}$$

and

$$\sigma_{m+} = \frac{1}{2}(\sigma_1 + \sigma_2)\,, \quad \tau_{m+} = \frac{1}{2}(\sigma_1 - \sigma_2) \quad \text{for} \;\; \sigma_1 \geq \sigma_3 \geq \sigma_2 \tag{15}$$

must be used. Then, the criterion reads:

$$\tau_{m-} = g(\sigma_{m-}) \quad \text{for} \;\; \sigma_2 \geq \sigma_1 \geq \sigma_3, \tag{16}$$

and

$$\tau_{m+} = g(\sigma_{m+}) \quad \text{for} \;\; \sigma_1 \geq \sigma_3 \geq \sigma_2. \tag{17}$$

Our task is then to rewrite $\tau_{m-} = g(\sigma_{m-})$ into the form of $\tau_m = g_-(\sigma_m)$ and $\tau_{m+} = g(\sigma_{m+})$ into $\tau_m = g_+(\sigma_m)$.

After the substitution of $\sigma_1 = \sigma_m + \tau_m$ and $\sigma_3 = \sigma_m - \tau_m$ into σ_{m-}, τ_{m-}, σ_{m+}, τ_{m+} we obtain the implicit relationships of σ_m and τ_m, which define the functions $\tau_m = g_-(\sigma_m)$, $\tau_m = g_+(\sigma_m)$:

$$\tau_m - (1-2k)\,\sigma_m = 2\,g\!\left(\frac{-\tau_m + (1+2k)\,\sigma_m}{2}\right), \tag{18}$$

which defines $\tau_m = g_-(\sigma_m)$, and

$$\tau_m + (1-2k)\,\sigma_m = 2\,g\!\left(\frac{\tau_m + (1+2k)\,\sigma_m}{2}\right), \tag{19}$$

which defines $\tau_m = g_+(\sigma_m)$, where $k = \nu$ for plane strain and $k = 0$ for plane stress.

For some simple functions $g(\sigma_m)$, the explicit formulae for $g_-(\sigma_m)$, $g_+(\sigma_m)$ can be found, otherwise we have to resort to numerical constructions.

Of special importance are the boundary lines between subregions:

$$\tau_m = b_-(\sigma_m) = -\,(1-2k)\,\sigma_m \tag{20}$$

(between the subregions for $g_-(\sigma_m)$ and $g(\sigma_m)$), and

$$\tau_m = b_+(\sigma_m) = (1-2k)\,\sigma_m \tag{21}$$

(between the subregions for $g(\sigma_m)$ and $g_+(\sigma_m)$). With (18) (or (19), respectively), it can be easily shown that if $\tau_m = g(\sigma_m)$ and $b_-(\sigma_m)$ (or $b_+(\sigma_m)$) intersect in a point, so in the same point intersects them the line $\tau_m = g_-(\sigma_m)$ (or $\tau_m = g_+(\sigma_m)$, respectively) (Fig. 4).

As the procedure of obtaining $\tau_m = g_-(\sigma_m)$ and $\tau_m = g_+(\sigma_m)$ can be quite labourious, the first step in any analysis of the relevance of the out-of-plane principal stress for our particular failure criterion should be the

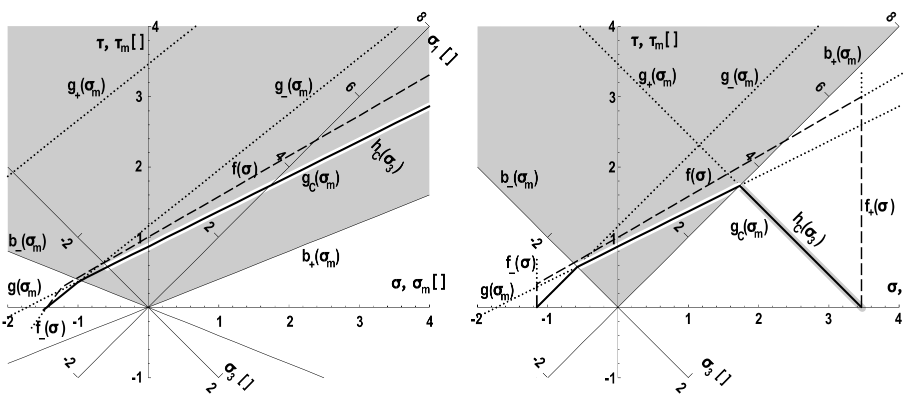

Fig. 4. The Coulomb criterion with out-of-plane stress taken into account: left (4a) in plane strain, right (4b) in plane stress. Grey is the formal validity region of the middle branch of (25), the portions of graphs beyond their validity regions are dotted, dashed are asymptotes of the Mohr envelope (26), (27). White or grey underline shows where the criterion $\tau_m = g_C(\sigma_m)$ or $\sigma_1 = h_C(\sigma_3)$ is physically relevant (to shear failure).

examination of whether and where does $\tau_m = g(\sigma_m)$ intersect $b_-(\sigma_m)$ and $b_+(\sigma_m)$. The lines $g(\sigma_m)$ and $b_-(\sigma_m)$ do always intersect (provided $g(\sigma_m)$ is defined for tensile stresses as well). As $g(\sigma_m)$ is a monotonously increasing function, and $g(0) > 0$, the sufficient condition for $g(\sigma_m)$ and $b_+(\sigma_m)$ to intersect in the region $\sigma_m \geq 0$ is

$$\lim_{s_m \to \infty} \frac{\mathrm{d}g}{\mathrm{d}\sigma_m}(s_m) < 1 - 2k, \tag{22}$$

where $k = \nu$ for plane strain and $k = 0$ for plane stress.

5. Coulomb criterion

Most often, we encounter the formulation of the Coulomb criterion in terms of the envelope of the Mohr circles, i.e., the formulation of type (5):

$$\tau = f(\sigma) = S_0 + \sigma\ \tan\phi, \tag{23}$$

where S_0 is the cohesive strength of the material, and ϕ is the angle of internal friction *(Jaeger and Cook, 1979)*.

For our purposes, we have to, using (8), turn it into a formula of type (6), which yields:

$$\tau_m = S_0 \cos\phi + \sigma_m \sin\phi. \tag{24}$$

Then we can proceed with the construction of the function $g_C(\sigma_m)$ (13). The implicit formulae (18), (19), can be easily arranged into explicit ones, so that in the end, the compound criterion $g_C(\sigma_m)$ reads:

$$\mathrm{g}_C(\sigma_\mathrm{m}) = \begin{cases} \dfrac{2S_0\cos\phi}{1+\sin\phi} + \left(1 - \dfrac{2k\,(1-\sin\phi)}{1+\sin\phi}\right)\sigma_m, & \tau_m \leq (-1+2k)\,\sigma_m\,, \\[2ex] S_0\cos\phi + \sigma_m\sin\phi, & \tau_m \geq (-1+2k)\,\sigma_m \wedge \tau_m \geq (1-2k)\,\sigma_m\,, \\[2ex] \dfrac{2S_0\cos\phi}{1-\sin\phi} - \left(1 - \dfrac{2k\,(1+\sin\phi)}{1-\sin\phi}\right)\sigma_m, & \tau_m \leq (1-2k)\,\sigma_m\,. \end{cases} \tag{25}$$

Using (9), we can construct the Mohr envelopes $f_-(\sigma)$, $f_+(\sigma)$ corresponding to $g_-(\sigma_m)$, $g_+(\sigma_m)$, respectively:

$$f_-(\sigma) = \frac{2S_0\cos\phi + ((1+2k)\sin\phi + (1-2k))\,\sigma}{2\sqrt{k\cos^2\phi - k^2(1-\sin\phi)^2}}, \tag{26}$$

$$f_+(\sigma) = \frac{2S_0\cos\phi + ((1+2k)\sin\phi - (1-2k))\,\sigma}{2\sqrt{k\cos^2\phi - k^2(1+\sin\phi)^2}}. \tag{27}$$

Nevertheless, let us be aware that the broken line of $f_-(\sigma)$, $f(\sigma)$, $f_+(\sigma)$ differs slightly (Fig. 4a) or considerably (Fig. 4b) from the true Mohr envelope corresponding to $g_C(\sigma_m)$ in the vicinity of "corners" of $g_C(\sigma_m)$. The envelope which respects the corners has to be constructed numerically.

The formula $\sigma_1 = h_C(\sigma_3)$ can be obtained easily:

$$\mathrm{h_C}(\sigma_3) = \begin{cases} \dfrac{2S_0\cos\phi + (1-k+(1+k)\sin\phi)\,\sigma_3}{k\,(1-\sin\phi)}, & \sigma_1 \le \dfrac{k\sigma_3}{1-k}, \\[2ex] \dfrac{2S_0\cos\phi + (1+\sin\phi)\,\sigma_3}{1-\sin\phi}, & \sigma_1 \ge \dfrac{k\sigma_3}{1-k} \wedge \sigma_1 \ge \dfrac{(1-k)\,\sigma_3}{k}, \\[2ex] \dfrac{2S_0\cos\phi + k\,(1+\sin\phi)\,\sigma_3}{1-\sin\phi - k\,(1+\sin\phi)}, & \sigma_1 \le \dfrac{(1-k)\,\sigma_3}{k}. \end{cases} \tag{28}$$

Into the illustrative examples to this criterion, we will input typical values of the Poisson ratio $\nu = 0.3$, the internal friction angle $\phi = \pi/6$, and the cohesive strength $S_0 = 1$, which can be scaled to any actual value in [Pa]. In the examples to all the other criteria, we will find equivalent material parameters, so that throughout the paper, the "rock" remains (almost) the same.

First, let us investigate the plane strain problem. After substitution of $k = \nu$, and the material parameter values given above, we can see that the intersection of $g_-(\sigma_m)$ and $g(\sigma_m)$ lies beyond the region of shear failure and that $g_+(\sigma_m)$ and $g(\sigma_m)$ do not intersect in the region of $\sigma_m \ge 0$. So, as (22) tells us, for $\phi > \arcsin(1-2\nu)$ (for $\nu = 0.3$, $\phi > 23.6°$), we do not need to consider the effect of out-of-plane stress in this case, at all.

In the plane stress problem ($k = 0$), the out-of-plane stress, however, plays a role, as can be seen from the plot in Fig. 4b.

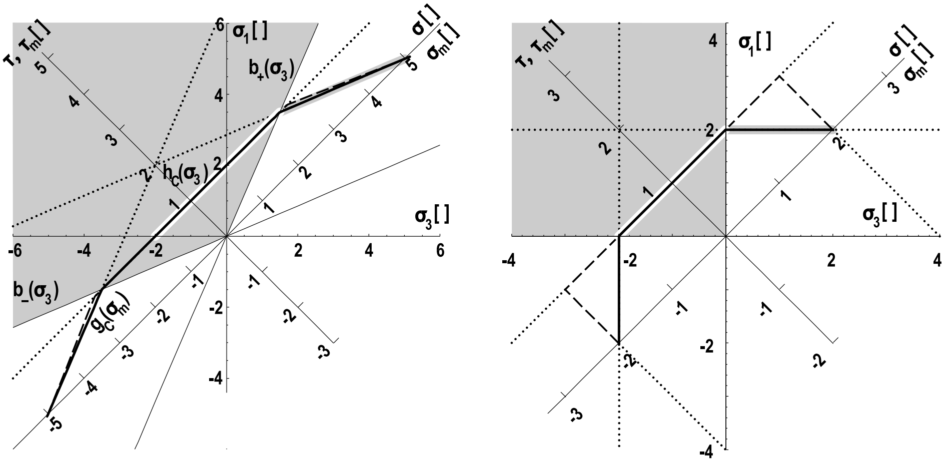

Fig. 5. The Tresca criterion with out-of-plane stress taken into account: left (5a) in plane strain, right (5b) in plane stress. Notice the change of axes compared to Fig. 4.

A special case of the Coulomb criterion for $\phi = 0$ is the Tresca criterion. The plots revealing the effects of the out-of-plane stress are shown in Fig. 5. Obviously, here we have to care about it even in the plane strain problem.

6. Hoek-Brown criterion

Most common is its formulation in terms of σ_1, σ_3. For our illustrative purposes, the original quadratic form of the criterion *(Hoek and Brown, 1980)* is the most suitable one:

$$\sigma_1 = h_{HB}(\sigma_3) = \sigma_3 + \sigma_{ci}\sqrt{m\frac{\sigma_3}{\sigma_{ci}} + s}. \tag{29}$$

Here σ_{ci} is the uniaxial compressive strength of the intact rock, m and s are material parameters, $s = 1$ for intact rock. First, let us find the least-squares Hoek-Brown approximation to the Coulomb criterion with the introduced tensile cut-off σ_t, $\sigma_t < 0$, i.e., to

$$h_{Ct}(\sigma_3) = \left(\frac{2S_0\cos\phi}{1-\sin\phi} + \frac{1+\sin\phi}{1-\sin\phi}\sigma_3\right) H(\sigma_3 - \sigma_t), \tag{30}$$

where $H(\sigma_3)$ is the unit step function. If we fix s $(s=1)$ and the same tensile cut-off σ_t for both functions (i.e., $h_{HB}(\sigma_t)=0$), then σ_{ci} can be eliminated:

$$\sigma_{ci}=\frac{2\sigma_t}{m\left(1-\sqrt{1+4s/m^2}\right)}, \tag{31}$$

and for $4s/m^2 \ll 1$ simplified to

$$\sigma_{ci}\approx -m\sigma_t/s. \tag{32}$$

Then, for the chosen upper bound $\sigma_{3\,\max}$, we obtain by variation of $\int\limits_{\sigma_t}^{\sigma_{3\,\max}}(h_{HB}(\sigma_3)-h_{Ct}(\sigma_3))^2\mathrm{d}\sigma_3$:

$$m=\frac{8\sqrt{s\left(1-\dfrac{\sigma_{3\,\max}}{\sigma_t}\right)}\,(5S_0\cos\phi+(3\sigma_{3\,\max}+2\sigma_t)\sin\phi)}{15\,(\sigma_{3\,\max}-\sigma_t)\,(1-\sin\phi)}\,. \tag{33}$$

The result of fitting for $\sigma_t=-0.5$, $\sigma_{3\,\max}=5$ ($m=7.28$, $\sigma_{ci}=3.64$) is shown in Fig. 6. For the sake of brevity, we will omit formulae of $g_C(\sigma_m)$, $h_C(\sigma_3)$ and present the results only graphically in Fig. 7.

7. Drucker-Prager criterion

The criteria (originally) used as criteria of yield of elasto-plastic materials are often (independently) used in brittle failure of elastic solids (and vice versa). This is the case of the Drucker-Prager criterion, as well (*Drucker and Prager, 1952, cf. Jaeger and Cook, 1979, p. 91, eq. 6*). We will respect the common practice of using this criterion in the new context under its old name (e.g. *Al-Ajmi, 2006*). As we can see from its formula,

$$\tau_{oct}=\kappa+\mu\sigma_{oct}, \tag{34}$$

where

$$\tau_{oct}=\frac{1}{3}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1-\sigma_3)^2+(\sigma_2-\sigma_3)^2}\,,$$

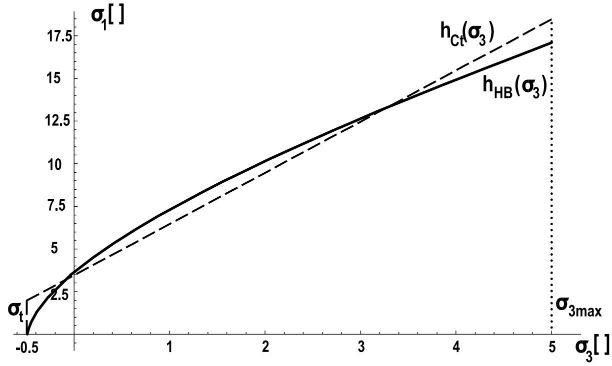

Fig. 6. The Hoek-Brown criterion fitting the Coulomb criterion with tensile cut-off.

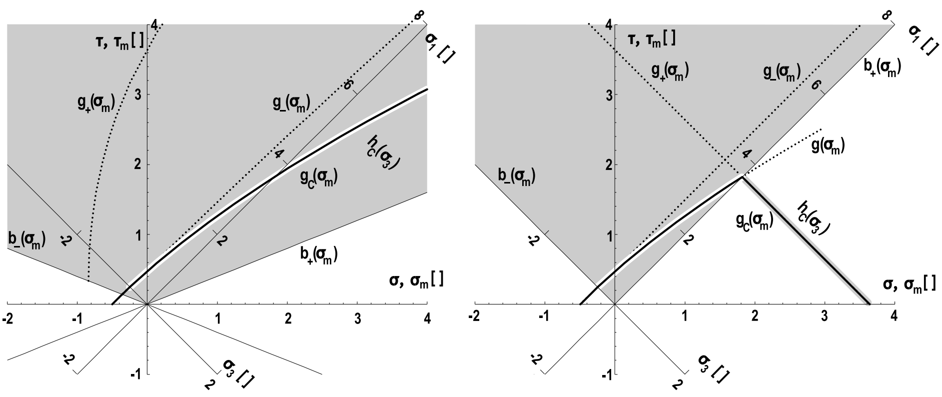

Fig. 7. The Hoek-Brown criterion with out-of-plane stress taken into account: left (7a) in plane strain, right (7b) in plane stress.

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3),$$

this three-dimensional criterion is completely insensitive to whether $\sigma_1 \geq \sigma_2 \geq \sigma_3$, $\sigma_2 \geq \sigma_1 \geq \sigma_3$ or $\sigma_1 \geq \sigma_3 \geq \sigma_2$. Thus, no piecewise function will complicate our lives. For $\sigma_2 = \sigma_3$, and for the parameter values:

$$\kappa = \frac{2\sqrt{2}S_0 \cos\phi}{3 - \sin\phi}, \quad \mu = \frac{2\sqrt{2}\sin\phi}{3 - \sin\phi}, \tag{35}$$

the criterion reduces to the Coulomb criterion (24), and the equivalents of $S_0 = 1$ and $\phi = \pi/6$ are then $\kappa = 0.98$ and $\mu = 0.57$.

The compound criterion $g_C(\sigma_m)$ for both planar problems can be obtained by substitution of $\sigma_2 = k(\sigma_1 + \sigma_3)$, $\sigma_1 = \sigma_m + \tau_m$, $\sigma_3 = \sigma_m - \tau_m$. The result is presented graphically in Fig. 8.

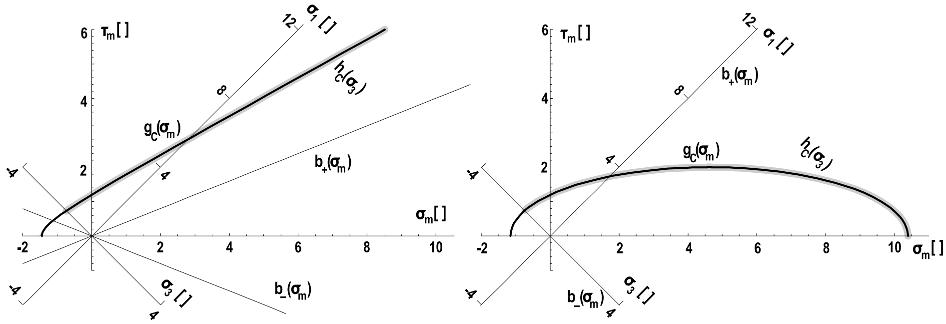

Fig. 8. The Drucker-Prager criterion with out-of-plane stress taken into account: left (8a) in plane strain, right (8b) in plane stress.

8. Linear Mogi criterion

The linear Mogi criterion *(Mogi, 1967)*, called the Mogi-Coulomb criterion in *Al-Ajmi (2006)*

$$\tau_{oct} = a + b\sigma_m, \tag{36}$$

where

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2},$$

$$\sigma_m = \frac{1}{2}(\sigma_1 + \sigma_3),$$

is similar to the Drucker-Prager, but has something in common with the two-dimensional criteria: the two principal stresses with the biggest difference determine the plane of failure, and the failure is inhibited by their mean.

Thus, as with the two-dimensional criteria, we have to construct for our planar problems a piecewise function (13) covering all the possible couples of principal stresses with the biggest difference. Again, we will leave the reader to do this task and present only the graphical outcome.

After the substitution of equivalent material parameters

$$a = \frac{2\sqrt{2}S_0\cos\phi}{3}, \quad b = \frac{2\sqrt{2}\sin\phi}{3}, \tag{37}$$

which are obtained analogically to (35), cf. *Al-Ajmi (2006)*, (numerical values $a = 0.82$, $b = 0.47$), we can see the surprising result: whereas in all other criteria, the compound criterion could have been constructed simply as

$$g_C(\sigma_m) = \min\Big\{g_-(\sigma_m),\ g(\sigma_m),\ g_+(\sigma_m)\Big\}, \tag{38}$$

where σ_m can take any value of the definition region of $g(\sigma_m)$, here this "alternative method" of construction fails – see the intersection of $g(\sigma_m)$ and $g_-(\sigma_m)$ (Fig. 9). This teaches us to adhere strictly to the reliable piecewise manner of construction (13).

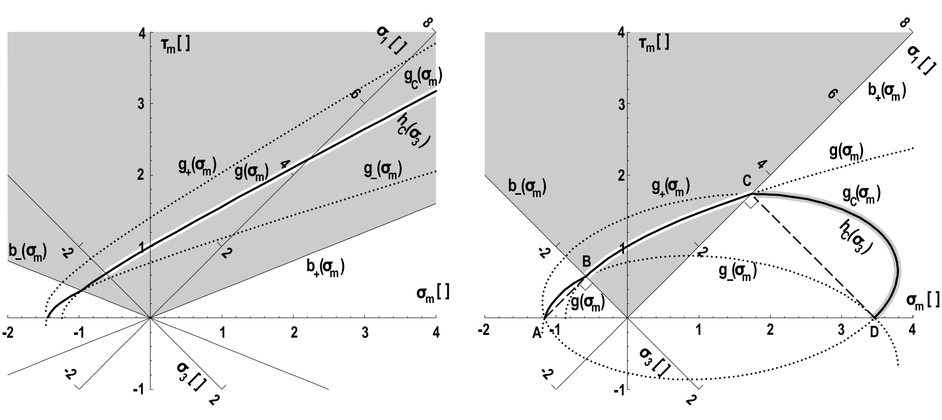

Fig. 9. The linear Mogi criterion with out-of-plane stress taken into account: left (9a) in plane strain, right (9b) in plane stress. Notice that the points A, B lie on the line $\sigma_3 = \sigma_t$, where $h(\sigma_t) = 0$ (dashed), and C, D on the line $\sigma_1 = h(0)$ (dashed) – similarly to the here presented two-dimensional criteria (cf. Fig 4b, 5b, 7b).

9. Conclusions

For typical geomaterials and for most of the failure criteria, the "threat" posed by forgetting to account for the out-of-plane stress in plane strain problem is negligible: in the plots of criteria, covering all possible values of principal stresses, the graphs $g(\sigma_m)$ and $b_-(\sigma_m)$ (or $b_+(\sigma_m)$, respectively) do not cross within the region of our interest, the exception being the Tresca criterion (which is not very realistic anyway).

The plots of the studied criteria in plane stress problems probably do not surprise anyone. It has to be said, however, that here the extension of the criteria into both the quadrants $\sigma_3 < 0 \ \wedge \ \sigma_1 < 0$ and $\sigma_3 > 0 \ \wedge \ \sigma_1 > 0$ is highly problematic: in the former, the actual mechanism of failure is not shear, in the latter, the real-world problem is not planar: if a thin plate or free surface is subject to coplanar compressive stresses, it has a tendency to bend and thus, present a problem of another category.

Moreover, our considerations are relevant only to purely planar problems, where the magnitude of out-of-plane principal stress is dictated solely by the magnitudes of the two in-plane principal stresses. In practice, however, quite frequent are the superimposed problems where the solution of the planar problem only modifies the "virgin" stress field with all the three principal stresses (independently) determined by tectonics, as it is in the case of boreholes (*Al-Ajmi, 2006*).

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