Rotating compositional and thermal convection in Earth's outer core

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Abstract: The linear stability of compositional and thermal convection in the Earth's rotating outer core was investigated. We have identified the values of Takens–Bogdanov bifurcation points and codimension-two bifurcation points by plotting graphs of neutral curves corresponding to stationary and oscillatory convection for different values of physical parameters relevant to rotating compositional and thermal convection in the Earth's outer core. We have also derived a nonlinear one–dimensional Landau–Ginzburg equation near the onset of stationary convection at a supercritical pitchfork bifurcation, and nonlinear one-dimensional coupled Landau–Ginzburg type equations near the onset of oscillatory convection at a supercritical Hopf bifurcation. We have also discussed the stability regions of standing and travelling waves.

Key words: rotating thermohaline convection, Earth's core, Landau-Ginzburg equation, standing and travelling waves

1. Introduction

Recent developments in both theoretical and experimental fluid dynamics have stimulated widespread interest in nonlinear fluid dynamical problems. Thermohaline convection, compositional, and simultaneously thermal convection, magnetoconvection and rotating thermal convection, etc., in the

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Earth's outer core are examples of double diffusive systems. In thermohaline convection, the temperature and the salt concentration provide two diffusive ingredients. In magnetoconvection, the temperature and the magnetic field provide two diffusivities.

Convection due to thermal and compositional buoyancy in the Earth's outer core is similar to the thermohaline convection, except for the fact that the compositional buoyancy in the core is due to the light ingredient (e.g. silicon) releasing at the inner/outer core boundary at solidification process, when the heavier iron only goes into the solid phase in the inner core. Therefore, the source of the outer core compositional buoyancy is lighter than the accompanying iron. Thus it is not like in thermohaline convection of salty water, where the salt, the source of buoyancy, is heavier than the water. The convection due to thermal and compositional buoyancy is often termed "thermohaline convection". However, e.g. the Earth's core conditions may be the reason to term it "thermosolutal convection" or "thermoconcentration convection". The latter is usually used in Russian literature.

In salty solutions temperature gradient can drive also mass currents and not only the thermal flows. This, the so called Soret effect, however, cannot be excluded in the Earth's outer core. Despite lacking experiments and experience corresponding to the core conditions, it would be heuristically successful to study the Soret effect influence on the Earth's outer core convection. A phenomenological relation between the mass current $\vec{j_C}$ and the local gradients can be written as

$$\vec{j_C} = -\left(D\nabla C + D\frac{K_T}{T_0}\nabla T\right), \tag{1}$$

where T is the temperature, K_T is the Soret coefficient which measures the cross coupling between temperature gradients and mass fluxes, therefore also called the thermodiffusion coefficient. K_T can be negative, as well as positive. D is the mass diffusivity, C is the concentration of the lighter component of the liquid. The mass conservation law

$$\partial_t C + \left(\vec{V}\cdot\nabla\right)C + \nabla\cdot\vec{j_C} = 0,$$

gives the diffusion equation for the concentration field in the form

$$\partial_t C + \left(\vec{V}\cdot\nabla\right)C = D\nabla^2 C + D\frac{K_T}{T_0}\nabla^2 T. \tag{2}$$

The temperature diffusion is not affected by the thermo-diffusion coefficient. The temperature diffusion equation then becomes

$$\partial_t T + \left(\vec{V} \cdot \nabla\right) T = \kappa \nabla^2 T. \tag{3}$$

Here κ is thermal diffusivity of the fluid, and we have a set $\partial_\eta = \partial / \partial\eta$ with $\eta = \{x,\ z,\ t,\ q,\ X,\ T\}$. The last term in Eq. (2) represents the Soret effect. *Hurle and Jakeman (1971)* have considered a nonlinear Soret effect. Owing to the two-component nature of the compositional and thermal convection in Earth's rotating outer core, one has the Soret effect which leads to the additional control parameter besides the Rayleigh number Ra, namely the separation ratio ψ. It is a measure of the stabilizing or destabilizing effect associated with concentration gradients. *Schöpf and Zimmerman (1993)* have studied the near-threshold behaviour for thermal convection in a binary liquid heated from below for realistic boundary conditions at the top and bottom. Both compositional and thermal convections in Earth's rotating outer core, and Rayleigh–Benard convection in rotating fluid are capable of showing stationary convection at pitchfork bifurcation, oscillatory convection at Hopf bifurcation (both pitchfork and Hopf bifurcations are primary bifurcations), and stationary convection at Takens–Bogdanov bifurcation point and codimension two point (Takens–Bogdanov bifurcation point and codimension two point are secondary bifurcations). Takens–Bogdanov bifurcation point is one at which the neutral curve of oscillatory convection intersects the neutral curve of the stationary convection and the frequency on the neutral curve of oscillatory convection approaches zero. This Takens–Bogdanov bifurcation point (where Rayleigh number for oscillatory convection coincides with Rayleigh number for stationary convection at the same wave number) is different form codimension two point (where Rayleigh number for the onset of oscillatory convection coincides with Rayleigh number for the onset of stationary convection, but at different wave numbers). The onset of instabilities in rotating thermohaline convection has been considered by *Pearlstein (1981)*, and in rotating magnetoconvection by *Tagare (1997)*, *Tagare and Rameshwar (2003)*. These rotating double-diffusive convective systems are expected to show a curve of Takens–Bogdanov bifurcation points which cumulates into a tertiary bifurcation point (corresponding to a triple zero eigenvalue). The problem, where the Taylor number is chosen so that there is a triple zero eigenvalue has been investigated for rotating thermohaline convection by *Arneodo et al. (1985)*.

In Section 2, we write basic equations of compositional and thermal convection in Earth's rotating outer core. In Section 3, we study the linear stability analysis. Since the bifurcation is a continuous one, only a slow modulation of the convective roll pattern is allowed by the fluid equations near the onset. The time evolution of general pattern is developed in Section 4 for a region $|\psi| > |\psi^*|$(where $\psi = \psi^*$ corresponds to a critical value of separation parameter in rotating compositional and thermal convection at a Takens–Bogdanov bifurcation point) by means of multiple scale analysis developed by *Newell and Whitehead (1969)* and *Segel (1969)* for a weakly nonlinear case. In a weakly nonlinear analysis performed in Section 4 and Section 5, a small amplitude convection cell is imposed on the basic flow. If this amplitude is of $O(\epsilon)$, then the interaction of the cell with itself forces a second harmonic and a mean state of correction of $O(\epsilon^2)$, and these in turn drives an $O(\epsilon^3)$ correction to the fundamental component of the imposed roll. A solvability criterion for this last correction yields an equation for the scale amplitude (which is called Landau–Ginzburg equation). In Section 4, we derive a nonlinear one-dimensional, time-dependent Landau–Ginzburg equation in complex amplitude, $A(X, T)$ with real coefficients near a supercritical pitchfork bifurcation. The phase of the complex amplitude $A(X, T)$ describes changes in the position and direction of rolls and its magnitude modulates the intensity of the convective motion. In Section 5, we derive a nonlinear one dimensional, time dependent coupled Landau–Ginzburg type equations in complex amplitudes $A_{1R}\,(X, \tau, T)$ and $A_{1L}\,(X, \tau, T)$ with complex coefficients. Here A_{1R}, and A_{1L} stand for right hand and left hand travelling waves respectively. Following *Matthews and Rucklidge (1993)*, we have dropped slow space dependence and obtained ODE's, with complex coefficients, termed as Landau equations, and discussed the stability regions of travelling waves and standing waves. In Section 6, we give the conclusions of this paper.

2. Basic equations

Consider a horizontal layer of compositional and thermal convection in Earth's rotating outer core, of depth d with linear temperature and concentration (of the lighter component) gradients, which is kept rotating at a constant angular velocity Ω about the z-axis. Following *Bhattacharjee (1987)*, we have for density

$$\rho = \rho_0 \left[1 - \alpha_T \left(T - T_0\right) - \alpha_C \left(C - C_0\right)\right], \tag{4}$$

where ρ_0 is the mean density of the system, α_T is the thermal expansion coefficient and α_C describes how the density of the fluid in Earth's rotating outer core, changes with changing concentration of the lighter component. Here α_T is a positive constant and α_C is positive in Earth's rotating outer core. The temperature difference ΔT is related to the concentration difference ΔC under steady-state conditions by the relation

$$\Delta C = -\frac{K_T}{T_0}\Delta T + \text{constant}. \tag{5}$$

We use the Cartesian system of co-ordinates whose dimensionless vertical co-ordinate z and dimensionless horizontal co-ordinates x, y are scaled with d. The velocity vector $\vec{V}(u, v, w)$, the density ρ, the temperature θ (deviation from conductive state), the concentration C, the time t and the pressure P are non-dimensionalized by scales $\kappa/d,\ \rho_0,\ \Delta T,\ \Delta C, d^2/\kappa$ and $\rho_0\kappa^2/d^2$. In the Boussinesq approximation one considers the fluid incompressible, except when dealing with the buoyancy terms that drives the concentration. The dimensionless parameters required for the description of the motion are: Rayleigh number $Ra = g\alpha_T\Delta T d^3/\kappa\nu$, Taylor number $Ta = 4\Omega^2 d^4/\nu^2$, Prandtl number $Pr = \nu/\kappa$, Lewis number $L = D/\kappa$ and separation parameter $\psi = -K_T\alpha_C/T_0\alpha_T$. The basic dimensionless equations for rotating compositional and thermal convection in the Boussinesq approximation, are:

$$\nabla \cdot \vec{V} = 0, \tag{6}$$

$$\frac{1}{Pr}\left[\partial_t\vec{V} + \left(\vec{V}\cdot\nabla\right)\vec{V}\right] = -\ \nabla\left(P - \frac{PrTa}{8}\left|\widehat{\Omega}\times\vec{r}\right|^2\right) + \nabla^2\vec{V} + \\ + Ta^{\frac{1}{2}}\left(\vec{V}\times\widehat{\Omega}\right) + Ra\left(\theta + \psi C\right)\hat{\vec{e}}_z, \tag{7}$$

$$\partial_t\theta + \left(\vec{V}\cdot\nabla\right)\theta = w + \nabla^2\theta, \tag{8}$$

$$\frac{1}{L}\left[\partial_t C + \left(\vec{V}\cdot\nabla\right)C\right] = \frac{w}{L} + \nabla^2 C - \nabla^2\theta, \tag{9}$$

where $\widehat{\Omega}$ is a unit vector along the axis of rotation. Here we consider $\widehat{\Omega} = \hat{\vec{e}}_z$. Eqs. (6–9) can be reduced to a form

$$\mathcal{L}w = \mathcal{N}, \tag{10}$$

where

$$\mathcal{L} = \mathcal{D}_\kappa \mathcal{D}_C \left[\mathcal{D}_\nu^2 \nabla^2 + Ta\, \partial_z^2\right] - Ra\, \mathcal{D}_\nu \nabla_h^2 \left[(1+\psi)\, \mathcal{D}_C - \tfrac{\psi}{L}\nabla^2\right], \tag{11}$$

$$\begin{aligned}\mathcal{N} = Pr^{-1} \mathcal{D}_\kappa \mathcal{D}_C \left\{Ta^{\frac{1}{2}} \partial_z A_z + \mathcal{D}_\nu \hat{\vec{e}}_z \cdot \nabla \times \vec{A}\right\} - \\ - Ra\, \mathcal{D}_\nu \nabla_h^2 \left\{\left(\mathcal{D}_C + \psi \nabla^2\right)\left(\vec{V}\cdot\nabla\right)\theta + \frac{\psi}{L}\mathcal{D}_\kappa \left(\vec{V}\cdot\nabla\right) C\right\},\end{aligned} \tag{12}$$

and

$$\nabla^2 = \partial_x^2 + \partial_z^2, \quad \vec{A} = \left(\vec{V}\cdot\nabla\right)\vec{\omega} - \left(\vec{\omega}\cdot\nabla\right)\vec{V},$$
$$\mathcal{D}_\kappa = (\partial_t - \nabla^2), \quad \mathcal{D}_C = (\tfrac{1}{L}\partial_t - \nabla^2), \quad \mathcal{D}_\nu = (\tfrac{1}{Pr}\partial_t - \nabla^2).$$

3. Linear stability analysis

In this section we study the linear stability analysis of compositional and thermal convection in Earth's rotating outer core, by substituting $w = W(z)\mathrm{e}^{iq_x x + pt}$ into linearized version of Eq. (10)

$$\mathcal{L}w = 0,$$

resulting in

$$\begin{aligned}&(D^2 - q^2 - p)\left(D^2 - q^2 - \tfrac{p}{L}\right)\left[\left(D^2 - q^2\right)\left(D^2 - q^2 - \tfrac{p}{Pr}\right)^2 + TaD^2\right] W = \\ &= -Raq^2\left(D^2 - q^2 - \tfrac{p}{Pr}\right)\left[\left(D^2 - q^2 - \tfrac{p}{L}\right) + \psi\left(\left(D^2 - q^2\right)\left(1 + \tfrac{1}{L}\right) - \tfrac{p}{L}\right)\right] W.\end{aligned} \tag{13}$$

In this paper we have considered only the idealized stress-free conditions on the surface and vanishing of temperature and concentration fluctuations. Thus $W = D^2W = D^4W = 0$ at $z = 0,\ 1$. W and its even derivatives vanish at $z = 0$ and $z = 1$ implies that we can assume $W = \sin \pi z$ as a solution of Eq. (13). Substituting $W = \sin \pi z$ and ω, the frequency of oscillations using $p = i\omega$, in Eq. (13), we get

$$\frac{q^2}{K} R\,a = \delta^2 \left[\delta^8 \psi_L + \frac{\omega^2 \delta^4}{L^2} - \frac{\omega^2}{Pr} \left\{ \delta^4 \left(\psi_L + \frac{\psi}{L^2} \right) + \frac{\omega^2}{L^2} (1+\psi) \right\} \right] + \\ + \frac{Ta\pi^2}{\left(\delta^4 + \frac{\omega^2}{Pr^2}\right)} \left[\delta^8 \psi_L + \frac{\omega^2 \delta^4}{L^2} + \frac{\omega^2}{Pr} \left\{ \delta^4 \left(\psi_L + \frac{\psi}{L^2} \right) + \frac{\omega^2}{L^2} (1+\psi) \right\} \right] + \\ + i \frac{\omega\,\delta^2}{\left(\delta^4 + \frac{\omega^2}{Pr^2}\right)} \left(A_1 \omega^4 + A_2 \omega^2 + A_3 \right), \tag{14}$$

where

$$\delta^2 = \pi^2 + q^2, \quad \psi_L = 1 + \psi \left(1 + \frac{1}{L} \right), \quad K^{-1} = \delta^4 \psi_L^2 + \frac{\omega^2 (1+\psi)^2}{L^2},$$

$$A_1 = \frac{\delta^2}{Pr^2 L^2} \left(1 + \psi + \frac{1}{Pr} \right), \tag{15}$$

$$A_2 = \delta^6 \left[\frac{1}{Pr^2} \left\{ \left(1 + \frac{1}{Pr} \right) + \psi \left[\left(1 + \frac{1}{L} \right) \left(1 + \frac{1}{Pr} \right) + \frac{1}{L^2} \right] \right\} + \\ + \frac{1}{L^2} \left(1 + \psi + \frac{1}{Pr} \right) \right] + \frac{Ta\pi^2}{L^2} (1 + \psi - \frac{1}{Pr}), \tag{16}$$

$$A_3 = \delta^{10} \left\{ \left(1 + \frac{1}{Pr} \right) + \psi \left[\left(1 + \frac{1}{L} \right) \left(1 + \frac{1}{Pr} \right) + \frac{1}{L^2} \right] \right\} + \\ + Ta\pi^2 \delta^4 \left\{ \left(1 - \frac{1}{Pr} \right) + \psi \left[\left(1 + \frac{1}{L} \right) \left(1 - \frac{1}{Pr} \right) + \frac{1}{L^2} \right] \right\}. \tag{17}$$

From relation (15), $A_1 > 0$ for $\psi > -1 - \frac{1}{Pr}$. We consider the following two cases:

3.1. Stationary convection ($\omega = 0$):

Substituting $\omega = 0$ in (14), we get

$$Ra_s = \frac{\delta^6 + \pi^2 Ta}{q^2 \psi_L}. \tag{18}$$

Here Ra_s is the value of Ra for the stationary convection. The minimum value of Ra_s obtained for $q = q_{sc}$ where

$$2\left(\frac{q_{sc}}{\pi}\right)^6 + 3\left(\frac{q_{sc}}{\pi}\right)^4 = 1 + \frac{Ta}{\pi^4}. \tag{19}$$

The wave number is identical to that for the single component fluid, while the threshold for the onset of stationary convection at pitchfork bifurcation is given by Eq. (18) with $q = q_{sc}$. Thus

$$Ra_{sc} = \frac{\delta_{sc}^6 + \pi^2 Ta}{q_{sc}^2 \psi_L}. \tag{20}$$

3.2. Oscillatory convection ($\omega^2 > 0$):

For oscillatory convection $\omega \neq 0$ and from Eq. (14), Ra will be complex. But the physical meaning of Ra requires it to be real. The condition that Ra is real implies that the imaginary part of Eq. (14) is zero, i.e.,

$$A_1\omega^4 + A_2\omega^2 + A_3 = 0. \tag{21}$$

If $\psi > -\left(1 + \frac{1}{Pr}\right)$ then $A_1 > 0$. If

$$Ta > \frac{\delta^6 (1 + Pr)\left(\frac{1}{Pr^2} + \frac{1}{L^2}\right)}{L^2\pi^2 (1 - Pr)}, \tag{22}$$

and $Pr < 1$, then $A_2 > 0$ and $A_3 < 0$. In this case there is one real positive value of ω^2 corresponding to the oscillatory convection. Other value of ω^2 will be negative. Substituting $W = \sin \pi z$ in Eq. (13), we get a fourth degree polynomial equation in p of the form

$$a_4p^4 + a_3p^3 + a_2p^2 + a_1p + a_0 = 0, \tag{23}$$

where

$$a_4 = \frac{\delta^2}{LPr^2},$$
$$a_3 = \delta^4\left[\frac{1}{Pr^2} + \frac{1}{LPr^2} + \frac{2}{LPr}\right],$$

$$a_2 = \delta^6 \left[\frac{2}{Pr} + \frac{2}{LPr} + \frac{1}{Pr^2} + \frac{1}{L} \right] + \frac{\pi^2 Ta}{L} - \frac{Ra\, q^2}{LPr} \left(\psi + 1 \right),$$

$$a_1 = \delta^2 \left\{ \delta^6 \left(1 + \frac{1}{L} + \frac{2}{Pr} \right) + \left(1 + \frac{1}{L} \right) \pi^2 Ta - Ra\, q^2 \left[\left(\frac{1}{Pr} + \frac{1}{L} \right) + \right. \right.$$
$$\left. \left. + \psi \left(\frac{1}{L} + \frac{1}{Pr} + \frac{1}{LPr} \right) \right] \right\},$$

$$a_0 = \delta^4 \left[\delta^6 + \pi^2 Ta - Ra\, q^2 \psi_L \right]. \tag{24}$$

Setting $p = i\omega$ in Eq. (23) and equating its real and imaginary parts to zero, we get

$$a_4 \omega^4 - a_2 \omega^2 + a_0 = 0, \tag{25}$$

$$\left(a_3 \omega^2 - a_1 \right) \omega = 0. \tag{26}$$

From (23), if $\omega = 0$ then $a_0 = 0$ and we get stationary convection and Ra_s is determined by putting $Ra = Ra_s$ in $a_0 = 0$ (see also (18)). Thus $\omega = 0$ and $a_0 = 0$ are the conditions for the pitchfork bifurcation corresponding to stationary convection. From (26), we can have marginal stability if $\omega^2 = a_1 \,/\, a_3$ $(a_1 > 0)$ and

$$a_4 a_1^2 - a_1 a_2 a_3 + a_0 a_3^2 = 0. \tag{27}$$

In this case we get oscillatory convection and Ra_o (the value of Ra for the oscillatory convection) is obtained by putting $Ra = Ra_o$ in the expressions for $a_0,\ a_1,\ a_2,\ a_3,\ a_4$ of the set of Eqs. (24) into Eq. (27). Thus we get a quadratic equation in Ra_o.

The codimension two point is determined by the intersection of two lines $a_0 = 0$ and $a_1 a_4 - a_2 a_3 = 0$ under the condition $a_1 > 0$ in $(\psi,\ Ra)$–space. This corresponds to the simultaneous occurrence of pitchfork and Hopf bifurcation and quasiperiodic solutions of the system can be obtained in the nonlinear regime.

Takens–Bogdanov bifurcation point is determined by the intersection of the two curves $a_0 = 0$ and $a_1 = 0$ in $(\psi,\ Ra)$–space. Thus Takens–Bogdanov bifurcation point corresponds to a double zero eigenvalue of the linear growth rate.

At the codimension two point, we have

$$Ra_{sc}(q_{sc}) = Ra_{oc}(q_{oc}) \text{ but } q_{sc} \neq q_{oc}, \tag{28}$$

and at the Takens–Bogdanov bifurcation point, we have

$$Ra_s(q_s) = Ra_o(q_o) = Ra_c(q_c) \text{ and } q_s = q_o = q_c. \tag{29}$$

Eliminating Ra from $a_0 = a_1 = 0$, we get Takens–Bogdanov bifurcation point at

$$\psi = \psi^* = \frac{-\left[\delta_c^6\left(1+\frac{1}{Pr}\right)+\pi^2 Ta\left(1-\frac{1}{Pr}\right)\right]}{\delta_c^6\left[\left(1+\frac{1}{L}\right)\left(1+\frac{1}{Pr}\right)+\frac{1}{L^2}\right]+Ta\pi^2\left[\frac{1}{L^2}+\left(1+\frac{1}{L}\right)\left(1-\frac{1}{Pr}\right)\right]}. \tag{30}$$

From Eq. (30), ψ^* is always negative if $Pr \geq 1$.

The codimension two point is an intersection between the Hopf bifurcation and pitchfork bifurcation with distinct wave numbers in (ψ, Ra) plane.

At the Takens–Bogdanov bifurcation, the Hopf bifurcation and pitchfork bifurcation neutral curves intersect, and only a single wave number is present. Thus at the Takens–Bogdanov bifurcation point the oscillatory neutral curve intersects the stationary convection curve and the frequency on the oscillatory neutral curve approaches zero, as the intersection point is approached.

In Figs. 1–3, each solid line stands for stationary convection (pitchfork bifurcation), and dotted line stands for oscillatory convection (Hopf bifurcation). In these Figs. 1–3, we have shown the effect of several physical parameters, like Ta, Pr, L on the onset of both stationary convection and oscillatory convection. When a physical parameter increases keeping remaining parameters fixed, the onset of instabilities increases, i.e. the onset of stationary convection and oscillatory convection inhibits, when a parameter increases with the remaining parameters fixed. Figs. (2b) and (3b) show both primary bifurcations (pitchfork bifurcation and Hopf bifurcation) and secondary bifurcations (Takens–Bogdanov bifurcation point and co-dimension two bifurcation point). Figs. (4a), (4b) are plotted in (ψ, Ra) plane. Each solid and dotted line in Figs. (4a), (4b) represents the stationary convection and the oscillatory convection, respectively. In both figures we showed the effect of the Taylor number on the Takens–Bogdanov bifurcation point (Fig. 4a) and the co-dimension two point (Fig. 4b). In Fig. 4a, the intersection point of the solid and the dotted line corresponding to the fixed Taylor number, gives the Takens–Bogdanov bifurcation point. The Rayleigh

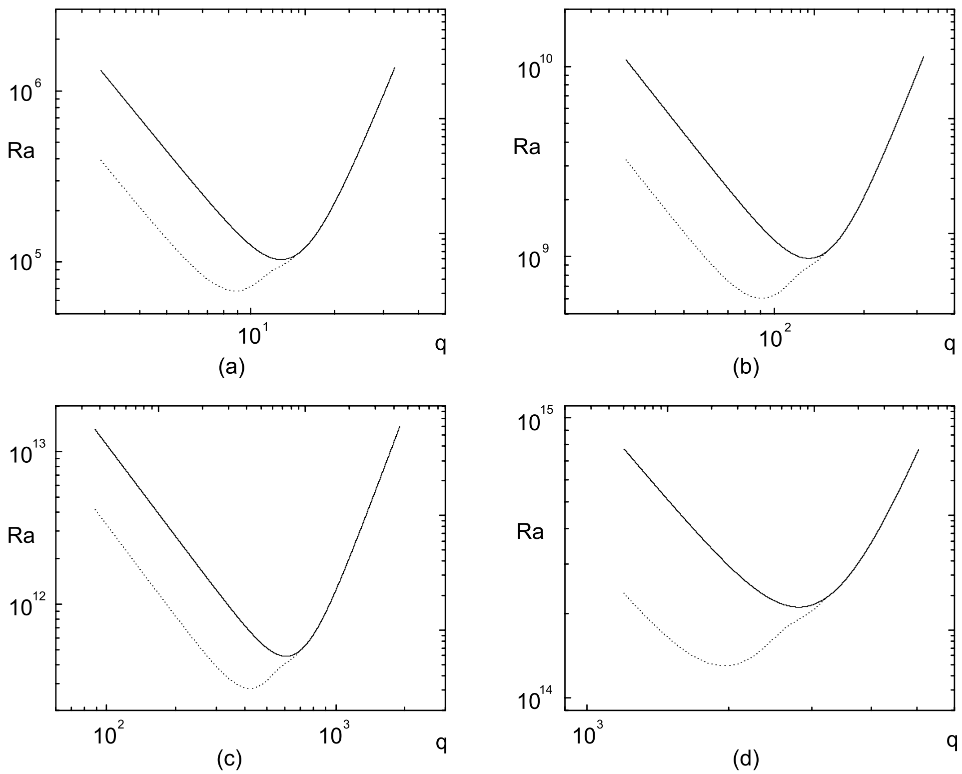

Fig. 1. Numerically calculated marginal stability curves (steady-solid lines, oscillatory-dotted lines) are plotted for $Pr = 0.5$, $L = 0.1$, $\psi = -0.01$ and (a) $Ta = 10^6$ (b) $Ta = 10^{12}$ (c) $Ta = 10^{16}$, (d) $Ta = 10^{20}$.

number Ra and the separation parameter ψ corresponding to the Takens–Bogdanov bifurcation point increases as Taylor number increases. We have $\psi = \psi^*$ at the Takens–Bogdanov. In the limit $\psi \longrightarrow \psi^*$, the frequency of the oscillatory instability tends to zero, and weakly nonlinear analysis in this region gives us a nonlinear equation describing the behavior of the system near the Takens–Bogdanov bifurcation. In Fig. 4b, critical Rayleigh numbers are taken on solid lines (stationary convection) and dotted lines (oscillatory convection) corresponding to Taylor number. In Fig. 4b, the point of intersection of solid line and dotted line corresponding to Taylor number is a co-dimension two point ($\omega \neq 0$). The critical Rayleigh number Ra and the separation parameter ψ corresponds to the co-dimension two

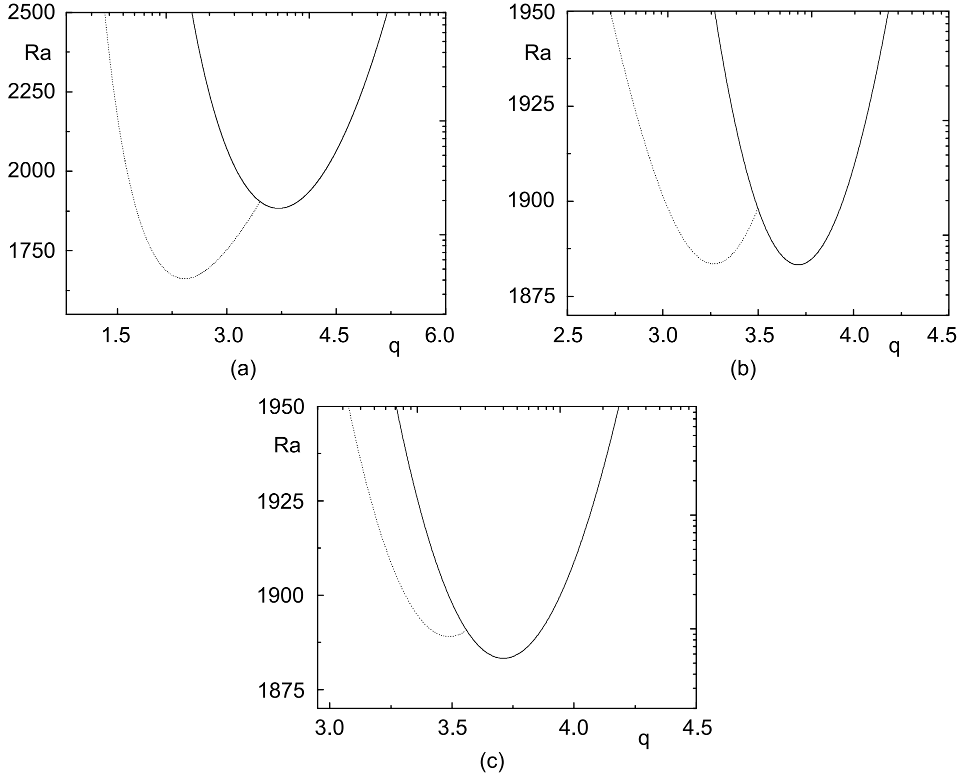

Fig. 2. Neutral curves for the stationary bifurcation (solid lines) and for the Hopf bifurcation (dashed lines) near the codimension two point for $Ta = 1000,\ L = 0.1,\ \psi = -0.01$, (a) $Pr = 0.03$, (b) $Pr = 0.366628$ (c) $Pr = 0.6$.

point increases as the Taylor number increases. At the codimension two point, let $\psi = \psi'$ for a Taylor number. If $\psi < \psi'$, we get first instability as oscillatory convection. If $\psi > \psi'$, then we get stationary convection as a first instability. For $a_0 = a_1 = a_2 = 0$ which gives $\psi = \psi^{**},\ \omega = 0$ is a triple zero eigenvalue. Thus at $\psi = \psi^{**}$, which corresponds to a tertiary bifurcation, we have stationary convection. $\psi = \psi^{**}$ satisfies a quadratic equation

$$\psi^2 + \mu_1\psi + \mu_2 = 0, \tag{31}$$

where μ_1 and μ_2 are functions of L and Pr. We get Eq. (31) by eliminating Ta and Ra from $a_0 = a_1 = a_2 = 0$.

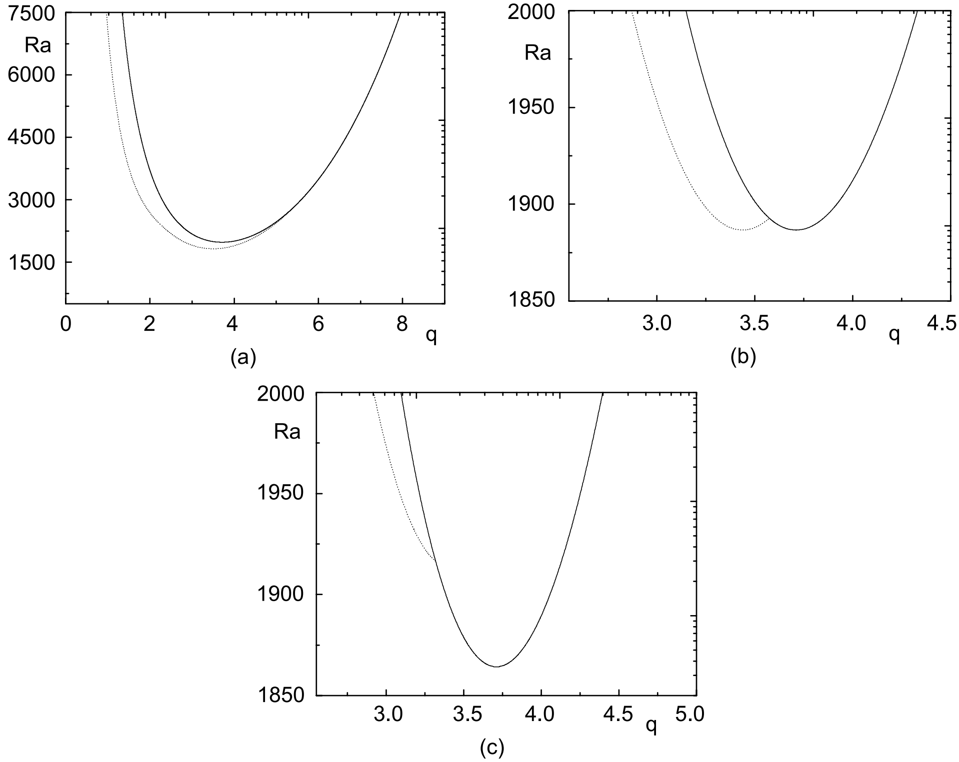

Fig. 3. Neutral curves for the stationary bifurcation (solid lines) and for the Hopf bifurcation (dashed lines) near the codimension two point for $Ta = 1000$, $Pr = 0.5$, $\psi = -0.01$, (a) $L = 0.07$, (b) $L = 0.0984348$ (c) $L = 0.11$.

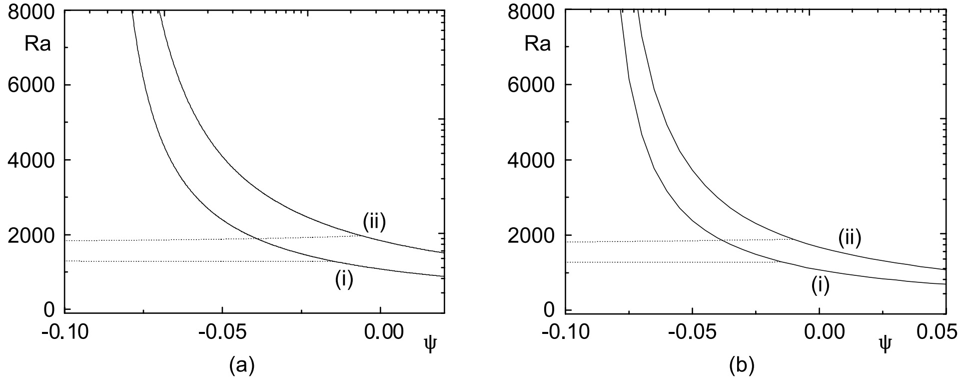

Fig. 4. Stability diagrams with $Pr = 0.5$, $L = 0.1$ (i) $Ta = 300$, (ii) $Ta = 1000$.

4. Onset of stationary convection at supercritical pitchfork bifurcation

The existence of a threshold (critical value of the Rayleigh number $Ra = Ra_{sc}$) and the cellular structure (critical wave number, $q = q_{sc}$) for a fixed Lewis number L, separation parameter ψ and Taylor number Ta are main characteristics of the stationary convection in the Earth's rotating outer core due to compositional and thermal buoyancy. In this section, we consider the region near the onset of the stationary convection by introducing ϵ as

$$\epsilon^2 = \frac{Ra_s - Ra_{sc}}{Ra_{sc}} \ll 1. \tag{32}$$

To simplify the problem, we assume the formation of rolls parallel to the axis y, i.e. the y-dependence disappears from Eq. (10). The z-dependence is contained entirely in the sin, cos functions which ensures that the free-free boundary conditions are satisfied. For values of the control parameter $Ra = Ra_s$ close to the threshold value $Ra_{sc}(\epsilon^2 \ll 1)$, we assume the solutions of Eqs. (6–9) in powers of ϵ:

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + ...,$$

where $f = (u, v, w, \theta, C)$ with the first approximation is given by the eigenvector of the linearized problem:

$$\begin{aligned}
u_0 &= \frac{i\pi}{q_{sc}} \left[A(X,T)e^{iq_{sc}x} \cos\pi z - \text{c.c.}\right], \\
v_0 &= -\frac{i\pi Ta^{\frac{1}{2}}}{\delta_{sc}^2 q_{sc}} \left[A(X,T)e^{iq_{sc}x} \cos\pi z - \text{c.c.}\right], \\
w_0 &= A(X,T)e^{iq_{sc}x} \sin\pi z + \text{c.c.}, \\
\theta_0 &= \frac{1}{\delta_{sc}^2} \left[A(X,T)e^{iq_{sc}x} \sin\pi z + \text{c.c.}\right], \\
C_0 &= \frac{\left(1+\frac{1}{L}\right)}{\delta_{sc}^2} \left[A(X,T)e^{iq_{sc}x} \sin\pi z + \text{c.c.}\right],
\end{aligned} \tag{33}$$

where $\delta_{sc}^2 = \pi^2 + q_{sc}^2$. Here c.c. stands for complex conjugate, $e^{iq_{sc}x} \sin\pi z$ is the critical mode for the linear problem at $Ra = Ra_{sc}$, and $q = q_{sc}$. The complex amplitude $A(X,T)$ depends on the slow variables. The independent variables $x,\ z,\ t$ are scaled by introducing multiple scales

$$X = \epsilon x,\ \ z = z,\ \ T = \epsilon^2 t, \tag{34}$$

and these formally separate the fast and slow dependent variables in f. The differential operators can be expressed as

$$\partial_x \longrightarrow \partial_x + \epsilon\partial_X, \quad \partial_z \longrightarrow \partial_z, \quad \partial_t \longrightarrow \epsilon^2\partial_T. \tag{35}$$

Using (35), the operators (11) and (12) can be written as

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \cdots, \\ \mathcal{N} &= \epsilon^2\mathcal{N}_0 + \epsilon^3\mathcal{N}_1 + \cdots, \end{aligned} \tag{36}$$

where

$$\mathcal{L}_0 = \nabla^4\left[\nabla^6 + Ta\partial_z^2 - Ra_{sc}\psi_L\partial_x^2\right], \tag{37}$$

$$\mathcal{L}_1 = 2\partial_x\partial_X\mathcal{L}_A, \tag{38}$$

$$\begin{aligned} \mathcal{L}_2 = & -\partial_T\nabla^2\left[\left(1+\frac{1}{L}+\frac{2}{Pr}\right)\nabla^6 + Ta\left(1+\frac{1}{L}\right)\partial_z^2 - \right. \\ & \left. -Ra_{sc}\partial_x^2\left\{\left(\frac{1}{L}+\frac{1}{Pr}\right)+\psi\left[\frac{1}{L}+\frac{1}{Pr}\left(1+\frac{1}{L}\right)\right]\right\}\right] + \partial_X^2\mathcal{L}_A + \\ & +4\partial_x^2\partial_X^2\left[10\nabla^6 + Ta\partial_z^2 - Ra_{sc}\psi_L\left(3\partial_x^2+2\partial_z^2\right)\right] - Ra_{sc}\psi_L\partial_x^2\nabla^4, \end{aligned} \tag{39}$$

where $\mathcal{L}_A = \nabla^2\left[5\nabla^6 + 2Ta\ \partial_z^2 - Ra_{sc}\psi_L\left(3\partial_x^2+\partial_z^2\right)\right]$. Using (32–35) in Eq. (10), and using definitions of $\mathcal{L}$ and $\mathcal{N}$ from (36), we get equating coefficients of various powers of ϵ to zero

$$\mathcal{L}_0 w_0 = 0, \tag{40}$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \tag{41}$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \tag{42}$$

Substituting the value of w_0 from (33) into (40) and using (37), we get

$$Ra_{sc} = \frac{\delta_{sc}^6 + Ta\,\pi^2}{q_{sc}^2\psi_L}. \tag{18}$$

Substituting the value of w_0 into $\mathcal{L}_1 w_0 = 0$, we get

$$2\left(\frac{q_{sc}}{\pi}\right)^6 + 3\left(\frac{q_{sc}}{\pi}\right)^4 = 1 + \frac{Ta}{\pi^4}. \tag{19}$$

Eq. (17), implies that $(\frac{\partial Ra_s}{\partial q_s})_{q_s=q_{sc}} = 0$. In Eq. (41), $\mathcal{N}_0 = 0$, $\mathcal{L}_1 w_0 = 0$ implies that Eq. (41) reduces to $w_1 = 0$. Similarly $u_1 = 0$,

$$\begin{aligned}
v_1 &= \frac{-i\pi^2 Ta^{\frac{1}{2}}}{4Pr q_{sc}^3 \delta_{sc}^2}\left[A^2 e^{2iq_{sc}x} - \text{c.c.}\right],\\
\theta_1 &= -\frac{1}{2\pi\delta_{sc}^2}|A|^2 \sin 2\pi z,\\
C_1 &= \frac{-(1+\frac{1}{L}+\frac{1}{L^2})}{2\pi\delta_{sc}^2}|A|^2 \sin 2\pi z = (1+\frac{1}{L}+\frac{1}{L^2})\,\theta_1.
\end{aligned} \tag{43}$$

Substituting zeroth order and first order solution in (42) and equating coefficient of $\sin\pi z$ in $\mathcal{N}_1 - \mathcal{L}_2 w_0$ to zero, we get

$$\lambda_0 \partial_T A - \lambda_1 \partial_X^2 A - \lambda_2 A + \lambda_3 |A|^2 A = 0, \tag{44}$$

where

$$\begin{aligned}
\lambda_0 &= \delta_{sc}^2\left\{\left(1+\frac{1}{L}+\frac{2}{Pr}\right)\delta_{sc}^6 + Ta\pi^2\left(1+\frac{1}{L}\right) - Ra_{sc}q_{sc}^2\left[\left(\frac{1}{L}+\frac{1}{Pr}\right)+\right.\right.\\
&\quad \left.\left.+\psi\left\{\frac{1}{L}+\frac{1}{Pr}\left(1+\frac{1}{L}\right)\right\}\right]\right\},\\
\lambda_1 &= 4q_{sc}^2\left\{10\delta_{sc}^6 + Ta\pi^2 - Ra_{sc}\psi_L\left(3q_{sc}^2 + 2\pi^2\right)\right\},\\
\lambda_2 &= Ra_{sc}q_{sc}^2\delta_{sc}^2\psi_L,\\
\lambda_3 &= -\frac{Ta\pi^4\delta_{sc}^2}{2Pr^2 q_{sc}^2} - \frac{Ra_{sc}q_{sc}^2\delta_{sc}^2}{2}\left\{1-\psi\left[1-\left(\frac{1}{L}+\frac{1}{L^2}+\frac{1}{L^3}\right)\right]\right\}.
\end{aligned} \tag{45}$$

Eq. (44) is called the Landau–Ginzburg equation, and it is meaningful only if λ_0, λ_1, λ_2 and λ_3 are positive. $\lambda_0 = 0$ at $\psi = \psi^*$. This implies that at the Takens–Bogdanov bifurcation point we have to use different scaling, and we will get partial differential equation with the second order derivative in time. λ_0 is always positive if $|\psi| > |\psi^*|$. If $\psi > 0$ then λ_2 is always positive. If $\psi < 0$ then λ_2 is positive only if

$$0 < \psi < \psi_c = -\frac{1}{\left(1+\frac{1}{L}\right)}. \tag{46}$$

Here we consider $\lambda_2 > 0$. λ_3 is positive if

$$Ra_{sc} > \frac{Ta\pi^4\delta_{sc}^2}{Pr^2 q_{sc}^4\left\{\delta_{sc}^2 + \psi\left[\delta_{sc}^2\left(1+\frac{1}{L}+\frac{1}{L^2}\right)-1\right]\right\}}. \tag{47}$$

This Landau–Ginzburg equation is valid only for $\lambda_3 > 0$ (supercritical bifurcation). $\lambda_3 = 0$ gives the tricritical bifurcation point. λ_3 changes its sign at the tricritical point (see Fig. 5). λ_1 is positive if the Taylor number satisfies

$$Ta > \frac{\delta_{sc}^4\left(2\pi^2 - 7q_{sc}^2\right)}{2\pi^2}. \tag{48}$$

By using the scaling (34) and $A(x,t) = A(X,T)/\epsilon$, Eq. (44) can be written in fast variables as

$$\lambda_0\partial_t A - \lambda_1\partial_x^2 A - \epsilon^2\lambda_2 A + \lambda_3|A|^2 A = 0. \tag{49}$$

Dropping the time dependence from Eq. (49), we get

$$\frac{d^2A}{dX^2} + \frac{\epsilon^2\lambda_2}{\lambda_1}(1 - \frac{\lambda_3}{\epsilon^2\lambda_2}|A|^2)A = 0. \tag{50}$$

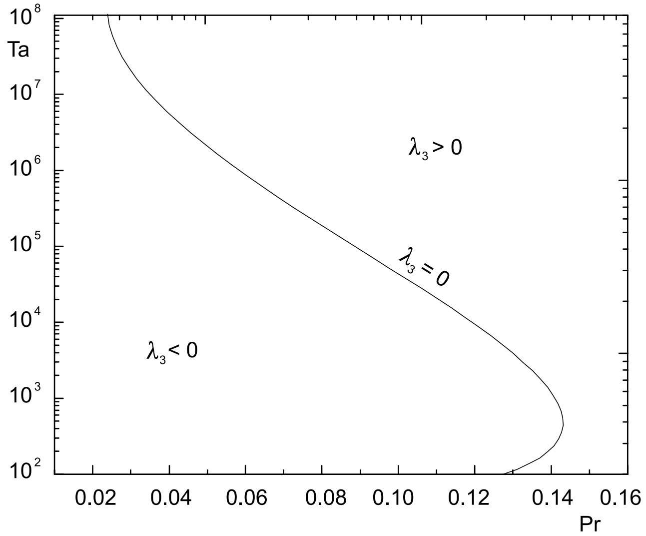

Fig. 5. $L = 0.1$, $\psi = -0.01$. λ_3 is the nonlinear coefficient of Landau-Ginzburg equation at the onset of stationary convection. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$ and subcritical if $\lambda_3 < 0$.

The solution of Eq. (50) is given by

$$A(X) = A_0 \tanh\left(\frac{X}{\Lambda}\right), \tag{51}$$

where

$$A_0 = \sqrt{\frac{\epsilon^2\lambda_2}{\lambda_3}} \text{ and } \Lambda = \sqrt{\frac{2\lambda_1}{\epsilon^2\lambda_2}}. \tag{52}$$

If we consider the system with finite aspect ratio, and if l is the characteristic length of the system, then we have

$$Ra(l) = Ra_b(q_b) = Ra_{sc}(q_{sc}) + \frac{\lambda_1 Ra_{sc}(q_{sc})\pi^2}{\lambda_2 l^2} + \cdots. \tag{53}$$

Here Ra_{sc} is the critical Rayleigh number of the system corresponding to $l \to \infty$ (q_b becomes q_{sc}) and $Ra(l) = Ra_b(q_b)$ is the critical Rayleigh number of the system of finite length in the horizontal direction, and $Ra_b(q_b)$ has a minimum value for $q = q_b$. Thus for $\lambda_1,\ \lambda_2 > 0$ finite l inhibits the onset of convection.

5. Derivation of Landau-Ginzburg type equations at the onset of oscillatory convection

In this section we treat the region near the onset of the oscillatory convection. We recast the hydrodynamic equations, and use the perturbation theory (multiple scale perturbation theory) in the manner of *Newell and Whitehead (1969)*. We now consider the Rayleigh number slightly above the critical value, i.e.,

$$Ra = Ra_{oc}\left(1 + \epsilon^2\right),$$

where $\epsilon \ll 1$. We write the solution of (6–9) in the power series of ϵ given as follows

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + \cdots, \tag{54}$$

where

$$f = f\left(u,\ v,\ w,\ \omega_x,\ \omega_y,\ \omega_z,\ \theta,\ C\right),$$

with first approximation is given by

$$u_0 = \frac{i\pi}{q_{oc}} \left[A_{1L} e^{i(q_{oc}x+\omega_{oc}t)} + A_{1R} e^{i(q_{oc}x-\omega_{oc}t)} - c.c. \right] \cos\pi z,$$

$$v_0 = -\frac{Ta^{\frac{1}{2}} i\pi}{q_{oc}} \left[\frac{A_{1L} e^{i(q_{oc}x+\omega_{oc}t)}}{\delta_{oc}^2 + \frac{i\omega}{Pr}} + \frac{A_{1R} e^{i(q_{oc}x-\omega_{oc}t)}}{\delta_{oc}^2 - \frac{i\omega}{Pr}} - c.c. \right] \cos\pi z,$$

$$w_0 = \left[A_{1L} e^{i(q_{oc}x+\omega_{oc}t)} + A_{1R} e^{i(q_{oc}x-\omega_{oc}t)} + c.c \right] \sin\pi z,$$

$$\omega_{x_0} = -\frac{Ta^{\frac{1}{2}} i\pi^2}{q_{oc}} \left[\frac{A_{1L} e^{i(q_{oc}x+\omega_{oc}t)}}{\delta_{oc}^2 + \frac{i\omega}{Pr}} + \frac{A_{1R} e^{i(q_{oc}x-\omega_{oc}t)}}{\delta_{oc}^2 - \frac{i\omega}{Pr}} - c.c. \right] \sin\pi z,$$

$$\omega_{y_0} = \frac{-i\delta_{oc}^2}{q_{oc}} \left[A_{1L} e^{i(q_{oc}x+\omega_{oc}t)} + A_{1R} e^{i(q_{oc}x-\omega_{oc}t)} - c.c. \right] \sin\pi z,$$

$$\omega_{z_0} = Ta^{\frac{1}{2}} \pi \left[\frac{A_{1L} e^{i(q_{oc}x+\omega_{oc}t)}}{\delta_{oc}^2 + \frac{i\omega}{Pr}} + \frac{A_{1R} e^{i(q_{oc}x-\omega_{oc}t)}}{\delta_{oc}^2 - \frac{i\omega}{Pr}} + c.c. \right] \cos\pi z,$$

$$\theta_0 = \left[\frac{A_{1L} e^{i(q_{oc}x+\omega_{oc}t)}}{\delta_{oc}^2 + i\omega} + \frac{A_{1R} e^{i(q_{oc}x-\omega_{oc}t)}}{\delta_{oc}^2 - i\omega} + c.c. \right] \sin\pi z,$$

$$C_0 = \left[h_1 A_{1L} e^{i(q_{oc}x+\omega_{oc}t)} + h_1^* A_{1R} e^{i(q_{oc}x-\omega_{oc}t)} + c.c. \right] \sin\pi z,$$

where

$$h_1 = \frac{\left(\frac{\delta_{oc}^2}{\delta_{oc}^2 + i\omega_{oc}} + \frac{1}{L} \right)}{\delta_{oc}^2 + \frac{i\omega_{oc}}{L}}, \quad h_1^* = \frac{\left(\frac{\delta_{oc}^2}{\delta_{oc}^2 - i\omega_{oc}} + \frac{1}{L} \right)}{\delta_{oc}^2 - \frac{i\omega_{oc}}{L}},$$

$\delta_{oc}^2 = \pi^2 + q_{oc}^2$ and *c.c.* stands for complex conjugate. Here A_{1L} denotes the amplitude of the left travelling wave of the roll, and A_{1R} denotes the amplitude of the right travelling wave of the roll, which are dependent on slow space and time variables (*Knobloch and De Luca, 1990*)

$$X = \epsilon x, \quad \tau = \epsilon t, \quad T = \epsilon^2 t, \tag{55}$$

and assume that $A_{1L} = A_{1L}(X, \tau, T),\ A_{1R} = A_{1R}(X, \tau, T)$. The differential operators $\partial_x,\ \partial_z$ and ∂_t, written as

$$\partial_x \longrightarrow \partial_x + \epsilon\partial_X, \quad \partial_z \longrightarrow \partial_z, \quad \partial_t \longrightarrow \partial_t + \epsilon\partial_\tau + \epsilon^2\partial_T. \tag{56}$$

The linear operator $\mathcal{L}$ and nonlinear operator $\mathcal{N}$ of Eq. (10) can be written by using Eq. (57) as

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \cdots, \tag{57}$$

$$\mathcal{N} = \epsilon^2 \mathcal{N}_0 + \epsilon^3 \mathcal{N}_1 + \cdots. \tag{58}$$

Using Eqs. (54, 57) and (58) in Eq. (10), we get by equating the coefficients of $\epsilon,\ \epsilon^2,\ \epsilon^3$

$$\mathcal{L}_0 w_0 = 0, \tag{59}$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \tag{60}$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \tag{61}$$

Eq. (59) is a linear problem. We get the critical Rayleigh number for the onset of oscillatory convection by using zeroth order solution w_0 into (59). At $O\left(\epsilon^2\right)$, $\mathcal{N}_0 = 0$ and $\mathcal{L}_1 w_0 = 0$ gives

$$\partial_\tau A_{1L} - \nu_g \partial_X A_{1L} = 0 \ \text{ and } \ \partial_\tau A_{1R} + \nu_g \partial_X A_{1R} = 0, \tag{62}$$

where $\nu_g = \partial_q \omega$ at $q = q_{oc}$ is the group velocity which is real. Hence from Eq. (60) we get $w_1 = 0$. From the equation of continuity we find that $u_1 = 0$. The relevant first order equations for $\omega_{z_1},\ \theta_1$ and C_1 are

$$\left(\frac{1}{Pr}\partial_t - \nabla^2\right)\omega_{z_1} = Ta^{\frac{1}{2}}\partial_z w_1 - \frac{1}{Pr}\left[\left(\vec{V_0}\cdot\nabla\right)\omega_{z_0} - \left(\vec{\omega_0}\cdot\nabla\right)w_0\right], \tag{63}$$

$$\left(\partial_t - \nabla^2\right)\theta_1 = Ra_{sc} w_1 - \left(\vec{V_0}\cdot\nabla\right)\theta_0, \tag{64}$$

$$\left(\frac{1}{L}\partial_t - \nabla^2\right)C_1 = \frac{w_1}{L} - \nabla^2\theta_1 - \frac{1}{L}\left(\vec{V_0}\cdot\nabla\right)C_0. \tag{65}$$

By using zeroth order solutions in Eqs. (63, 64) and (65), we get

$$\omega_{z_1} = \frac{Ta^{\frac{1}{2}}\pi^2}{Pr}\Bigg[h_2 A_{1L}^2 e^{2i(q_{oc}x+\omega_{oc}t)} + h_2^* A_{1R}^2 e^{2i(q_{oc}x-\omega_{oc}t)} + + \frac{\delta_{oc}^2}{q_{oc}^2}\left(\delta_{oc}^4 + \frac{\omega_{oc}^2}{Pr^2}\right)^{-1} A_{1L}A_{1R}e^{2iq_{oc}x} + c.c.\Bigg],$$

$$\omega_{x_1} = 0, \qquad \omega_{y_1} = 0,$$

$$v_1 = -\frac{Ta^{\frac{1}{2}} i\pi^2}{2q_{oc}Pr}\Bigg[h_2 A_{1L}^2 e^{2i(q_{oc}x+\omega_{oc}t)} + h_2^* A_{1R}^2 e^{2i(q_{oc}x-\omega_{oc}t)} +$$

$$+\frac{\delta_{oc}^2}{q_{oc}^2}\left(\delta_{oc}^4+\frac{\omega_{oc}^2}{Pr^2}\right)^{-1}A_{1L}A_{1R}e^{2iq_{oc}x}-c.c.\Bigg], \qquad (66)$$

$$\theta_1=-\pi\Bigg[\left(|A_{1L}|^2+|A_{1R}|^2\right)\frac{\delta_{oc}^2}{2\pi^2\left(\delta_{oc}^4+\omega_{oc}^2\right)}+$$
$$+h_3A_{1L}A_{1R}^*e^{2i\omega_{oc}t}+c.c.\Bigg]\sin 2\pi z,$$

$$C_1=-\Bigg\{\frac{1}{4\pi}\left[\frac{(h_1+h_1^*)}{L}+\frac{2\delta_{oc}^2}{\delta_{oc}^4+\omega_{oc}^2}\right]\left(|A_{1L}|^2+|A_{1R}|^2\right)+$$
$$+\pi\left(\frac{h_1}{L}+4\pi^2h_3\right)\left(2\pi^2+\frac{i\omega_{oc}}{L}\right)^{-1}A_{1L}A_{1R}^*e^{2i\omega_{oc}t}+c.c.\Bigg\}\sin 2\pi z,$$

$$h_2=\left[\left(\delta_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\left(2q_{oc}^2+\frac{i\omega_{oc}}{Pr}\right)\right]^{-1},$$
$$h_3=\left[\left(\delta_{oc}^2+i\omega_{oc}\right)\left(2\pi^2+i\omega_{oc}\right)\right]^{-1}.$$

Eq. (61) is solvable when $\mathcal{L}_0w_0=0$, one requires that its right hand side be orthogonal to w_0, which is ensured, if the coefficients of $\sin\pi z$ in $\mathcal{N}_1-\mathcal{L}_2w_0$ are equal to zero. This implies that

$$\Lambda_0\partial_TA_{1L}+\Lambda_1\left(\partial_\tau-\nu_g\partial_X\right)A_{2L}-\Lambda_2\partial_{X^2}A_{1L}-\Lambda_3A_{1L}+$$
$$+\Lambda_4|A_{1L}|^2A_{1L}+\Lambda_5|A_{1R}|^2A_{1L}=0, \qquad (67)$$
$$\Lambda_0\partial_TA_{1R}+\Lambda_1\left(\partial_\tau+\nu_g\partial_X\right)A_{2R}-\Lambda_2\partial_{X^2}A_{1R}-\Lambda_3A_{1R}+$$
$$+\Lambda_4|A_{1R}|^2A_{1R}+\Lambda_5|A_{1L}|^2A_{1R}=0. \qquad (68)$$

It should be noted that A_{1L}, A_{1R} are of order ϵ and A_{2L}, A_{2R} are of order ϵ^2. If $\omega_{oc}=0$ in $\Lambda_0,\Lambda_2,\Lambda_3$ and Λ_4 then the expressions match with the coefficients $\lambda_0,\lambda_1,\lambda_2,$ and λ_3 of Landau-Ginzburg equation at the onset of stationary convection.

From Eqs. (62), we get $A_{1L}(\xi',T)$ and $A_{1R}(\eta',T)$, where $\xi'=\nu_g\tau+X$, $\eta'=\nu_g\tau-X$. Eqs. (67, 68) can be written as

$$2\nu_g\Lambda_1\partial_{\eta'}A_{2L}=-\Lambda_0\partial_TA_{1L}+\Lambda_2\partial_{X^2}A_{1L}+\Lambda_3A_{1L}-$$
$$-\left(\Lambda_4|A_{1L}|^2+\Lambda_5|A_{1R}|^2\right)A_{1L}, \qquad (69)$$
$$2\nu_g\Lambda_1\partial_{\xi'}A_{2R}=-\Lambda_0\partial_TA_{1R}+\Lambda_2\partial_{X^2}A_{1R}+\Lambda_3A_{1R}-$$

$$- \left(\Lambda_4 \left| A_{1R} \right|^2 + \Lambda_5 \left| A_{1L} \right|^2 \right) A_{1R}. \tag{70}$$

Let $\xi' \in [0, l_1]$, $\eta' \in [0, l_2]$, where $l_1,\ l_2$ are periods of $A_{1L},\ A_{1R}$, respectively. Expansion (54) remains asymptotic for times $t = O(\epsilon^{-2})$ only if an appropriate solvability condition holds. This condition obtained by integrating Eq. (69) over η' and Eq. (70) over ξ', we get

$$\Lambda_0 \partial_T A_{1L} = \Lambda_2 \partial_{X^2} A_{1L} + \Lambda_3 A_{1L} - \left(\Lambda_4 \left| A_{1L} \right|^2 + \Lambda_5 \left| A_{1R} \right|^2 \right) A_{1L}, \tag{71}$$

$$\Lambda_0 \partial_T A_{1R} = \Lambda_2 \partial_{X^2} A_{1R} + \Lambda_3 A_{1R} - \left(\Lambda_4 \left| A_{1R} \right|^2 + \Lambda_5 \left| A_{1L} \right|^2 \right) A_{1R}. \tag{72}$$

The Eqs.(71, 72) are correct asymptotic evolution equations when $\nu_g = O(1)$.

5.1. Travelling wave and standing wave convection

To study the stability regions of travelling waves and standing waves we proceed as follows:

On dropping slow space variable X from Eqs. (71) and (72), we get a pair of first order ODE's

$$\frac{dA_{1L}}{dT} = \frac{\Lambda_3}{\Lambda_0} A_{1L} - \frac{\Lambda_4}{\Lambda_0} A_{1L} \left| A_{1L} \right|^2 - \frac{\Lambda_5}{\Lambda_0} A_{1L} \left| A_{1R} \right|^2, \tag{73}$$

$$\frac{dA_{1R}}{dT} = \frac{\Lambda_3}{\Lambda_0} A_{1R} - \frac{\Lambda_4}{\Lambda_0} A_{1R} \left| A_{1R} \right|^2 - \frac{\Lambda_5}{\Lambda_0} A_{1R} \left| A_{1L} \right|^2. \tag{74}$$

Put

$$\beta = \frac{\Lambda_3}{\Lambda_0}, \quad \gamma = -\frac{\Lambda_4}{\Lambda_0} \quad \text{and} \quad \delta = -\frac{\Lambda_5}{\Lambda_0}.$$

Then Eqs. (73) and (74) take the following form

$$\frac{dA_{1L}}{dT} = \beta A_{1L} + \gamma A_{1L} \left| A_{1L} \right|^2 + \delta A_{1L} \left| A_{1R} \right|^2, \tag{75}$$

$$\frac{dA_{1R}}{dT} = \beta A_{1R} + \gamma A_{1R} \left| A_{1R} \right|^2 + \delta A_{1R} \left| A_{1L} \right|^2. \tag{76}$$

Consider $A_{1L} = a_L e^{i\phi_L}$ and $A_{1R} = a_R e^{i\phi_R}$ (we can write a complex number in the amplitude and phase (angle) form), where

$$a_L = \left| A_{1L} \right|, \quad \phi_L = \arg(A_{1L}) = \tan^{-1} \left(\frac{\Im m(A_{1L})}{\Re e(A_{1L})} \right)$$

and

$$a_R = |A_{1R}|, \quad \phi_R = \arg(A_{1R}) = \tan^{-1}\left(\frac{\Im m(A_{1R})}{\Re e(A_{1R})}\right).$$

a_L, a_R, ϕ_L, and ϕ_R are functions of time T since A_{1L} and A_{1R} are functions of T. Thus a_L and a_R are positive functions.

Substituting the definitions of A_{1L} and A_{1R} and $\beta = \beta_1 + i\beta_2$, $\gamma = \gamma_1 + i\gamma_2$, $\delta = \delta_1 + i\delta_2$ into Eqs. (75) and (76), we get

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L |a_L|^2 + \delta_1 a_L |a_R|^2, \tag{77}$$

$$\frac{d\phi_L}{dT} = \beta_2 + \gamma_2 |a_L|^2 + \delta_2 |a_R|^2, \tag{78}$$

$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R |a_R|^2 + \delta_1 a_R |a_L|^2, \tag{79}$$

$$\frac{d\phi_R}{dT} = \beta_2 + \gamma_2 |a_R|^2 + \delta_2 |a_L|^2. \tag{80}$$

Eqs. (77) and (79) do not contain a phase term, so we take these two equations for the future discussions. We have Eqs. (77) and (79) as

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L^3 + \delta_1 a_L a_R^2,$$
$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R^3 + \delta_1 a_R a_L^2,$$

since a_L and a_R are positive functions. Put

$$\frac{da_L}{dT} = F_1(a_L, a_R), \qquad \frac{da_R}{dT} = F_2(a_L, a_R). \tag{81}$$

Now we discuss the stability of the equilibrium points of the above Eqs. (81). We get four equilibrium points like $(a_L, a_R) = (0, 0)$ [conduction state], $(a_L, a_R) = (a_L, 0)$ [a_L=amplitude of left travelling waves, here we get $F_2 = 0$, and we get one condition from $F_1 = 0$, i.e., $a_L^2 = -\frac{\beta_1}{\gamma_1}\left(= |A_{1L}|^2\right)$], $(a_L, a_R) = (0, a_R)$ [a_R = amplitude of right travelling waves, here $F_1 = 0$ and from $F_2 = 0$, we get $a_R^2 = -\frac{\beta_1}{\gamma_1}\left(= |A_{1R}|^2\right)$], and for $a_L \neq 0$ and $a_R \neq 0$ we get $(a_L, a_R) = \left(-\frac{\beta_1}{(\gamma_1+\delta_1)}, -\frac{\beta_1}{(\gamma_1+\delta_1)}\right)$ [this gives condition for standing waves]. At standing waves we have $A_{1L} = A_{1R}$, so $a_L = a_R$.

Now the Jacobian of F_1 and F_2 is given by

$$\begin{pmatrix} \frac{\partial F_1}{\partial a_L} & \frac{\partial F_1}{\partial a_R} \\ \frac{\partial F_2}{\partial a_L} & \frac{\partial F_2}{\partial a_R} \end{pmatrix}.$$

If real parts of all eigenvalues of the Jacobian are negative at an equilibrium point, then that point is a stable equilibrium [Lyapounov's theorem or principle of linearized stability]. Some valuable conditions for travelling waves and standing waves are: Travelling waves are stable if $\beta_1 > 0,\ \gamma_1 < 0$ and $\delta_1 < \gamma_1 < 0$. Standing waves are stable if $\beta_1 > 0,\ \gamma_1 < 0$ and (i) if $\delta_1 > 0,$ then $-\gamma_1 > \delta_1 > 0$, (ii) if $\delta_1 < 0,$ then $-\gamma_1 > -\delta_1 > 0$.

The problem of compositional and thermal convection in Earth's rotating outer core, with periodic boundary conditions, is studied by using a standard perturbation technique. Weakly nonlinear theory must be used to resolve which of the standing or travelling waves will occur at the onset of convection. The coefficients in Eq. (73) and (74) are complicated functions of the parameters $Ta,\ q_{oc},\ L,\ Pr$ and ψ, so it is not possible to give a simple criterion for the stability of the travelling and standing waves. The conditions for the travelling waves are given by $A_{1R} = 0,\ |A_{1L}|^2 = -\beta_1/\gamma_1$. The conditions for standing waves are given by $A_{1L} = A_{1R} \neq 0$, $|A_{1L}|^2 = |A_{1R}|^2 = -\beta_1/\left(\gamma_1 + \delta_1\right)$. For each set of parameter values, the linear problem was solved to determine whether stationary or oscillatory mode becomes unstable first, as Ra is increased. If it was found that the oscillatory mode became unstable, the coefficients $\Lambda_0,\ \Lambda_1,\ \Lambda_2,\ \Lambda_3,\ \Lambda_4$ were determined at the value of q_o that minimized Ra_o, to investigate the stability of travelling or standing waves. Figs. 6a-6c, show the results as a function of $Ta,\ \psi$ at $L = 0.1,\ Pr = 0.4,\ 0.1,\ 0.03$. In Fig. 6a, for $Pr = 0.4$, there is an intersection between travelling waves and standing waves. Let $\psi = \psi''$ at this intersection point. For some fixed value of $\psi < \psi''$, as Ta increases, we get initially the travelling waves, then the standing waves. For $\psi > \psi''$, for some fixed value of ψ, we get only standing waves. In Fig. 6b, for $Pr = 0.1$, for some fixed value of ψ, as Ta increases, we get initially the travelling waves, and then the standing waves. In Fig. 6c, for $Pr = 0.03$, there are repeated stability regions. For large values of Ta ($Ta > 10^{15}$), we get $\Re e(\gamma) = \Re e(\delta)$.

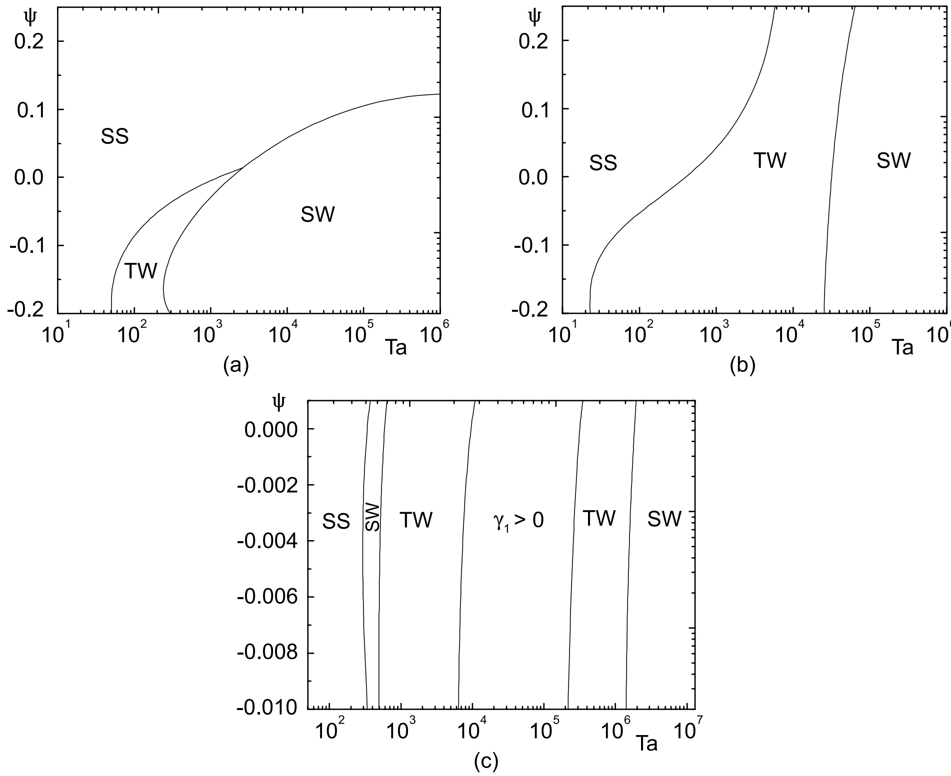

Fig. 6. Stability regions of stationary convection (SS), travelling waves (TW) and standing waves (SW), for $L = 0.1$, $\psi = -0.01$, (a) $Pr = 0.4$; (b) $Pr = 0.1$; (c) $Pr = 0.03$.

6. Conclusions

In this paper we have studied the compositional and thermal convection in the Earth's rotating outer core. The compositional and thermal convection in Earth's rotating outer core is an example of a double diffusive system, like the magnetoconvection and thermohaline convection. In fact, the thermohaline convection and the compositional and thermal convection in Earth's rotating outer core are examples of the double diffusive system, where density depends on two diffusive mechanisms, like $\rho = \rho_0[1 - \alpha(T - T_0) - \beta(C - C_0)]$, where we have thermal diffusivity and concentration or saline diffusivity in magnetoconvection; density does not depend on diffusive mechanism due to magnetic field. In this paper the stability of compositional and thermal convection in Earth's rotating outer core has been investigated. We have obtained the values of the Takens-Bogdanov bifurcation points and the codimension two points by plotting graphs of neutral curves corresponding to stationary convection and oscillatory convection for different values of physical parameters relevant to the compositional and thermal convection in Earth's rotating outer core. We have derived one dimensional Landau-Ginzburg equation at the onset of the supercritical pitchfork bifurcation and the one dimensional nonlinear coupled Landau-Ginzburg type equations at the onset of the supercritical Hopf bifurcation. We have also studied the stability regions of the travelling waves and standing waves in the (Ta, ψ) plane, and observed that when Pr decreases, then we get repeated stability regions (standing waves and travelling waves).

It is interesting to note that in the non-magnetic case with $Pr \to \infty$, $L \to 0$ and $\psi \to \infty$, the problem studied in this paper reduces to that of the compositional convection in the rotating mushy layers, as considered recently by *Guba and Bod'a (1998)* in the linear, and by *Guba (2001)* in the nonlinear regimes. The mushy layer is a region of coexisting liquid and solid phases, forming as a consequence of constitutional supercooling, when a binary alloy solidifies directionally (*Worster, 1997*). The rotational constraint, the effects of which are of particular interest in the present study, as well, was found to control the nature of the bifurcation to convection with both the oblique-roll planform and the planform of hexagonal symmetry. Thus, the results of the present study might be, in principle, used to draw qualitative conclusions regarding the nature of the Hopf bifurcation in mushy layers.

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