Evaluation of selected gravity field parameters from local high resolution gravity and elevation data

P. Novák
Research Institute of Geodesy, Topography and Cartography[1]
University of Western Bohemia, Department of Mathematics[2]

A b s t r a c t: Theory and numerical evaluation of selected parameters of the gravity field over a region in Central Europe, namely of the geoidal heights, deflections of the vertical and anomalous vertical gradients of gravity, are discussed in this manuscript. Input values for their numerical evaluation represent a detailed and accurate gravity and elevation database GOP30x30, that contains discrete values of mean gravity and elevation data on a homogeneous geographical grid with spacing of 30×30 arcsec, and the global geopotential model EGM96. Local gravity and elevation data are used for evaluation of high-frequency components of the sought parameters using discretized integral equations of Greens's kind. Discrete numerical integration is applied within a spherical cap centered at each computation point. The effect of gravity data outside the spherical cap is computed by the Molodensky approach using the spectral description of the global gravity field. The Stokes function is modified according to *Vaníček and Kleusberg (1987)* to minimize the effect of gravity data outside the spherical cap. The Vening-Meinesz function and the integration function in the gravity gradient integral are not modified due to their relatively fast attenuation with the increasing spherical distance. The low-frequency components of the same parameters are synthesized using the EGM96 spherical harmonic coefficients. Obtained results can be used in geodesy and geophysics.

Key words: geoid, deflections of the vertical, anomalous gravity gradient, Central Europe

1. Introduction

Among major tasks of contemporary geodesy an important role plays global gravity field mapping. New geopotential models are being solved

[1] Zdiby 98, 250 66 Czech Republic; e-mail: pnovak@pecny.asu.cas.cz
[2] Univerzitní 8, 306 14 Pilsen, Czech Republic; e-mail: panovak@kma.zcu.cz

through analyzing abundant observation material provided namely by the satellite missions CHAMP and GRACE. Despite significant advances in the global gravity field modelling, it is still inevitable to use also local ground gravity and elevation data since only they can provide the information on the medium and high-frequency spectrum of different gravity field parameters. Often local gravity data are combined with the global information in order to take advantage of their respective spectral properties. This approach is also used in this contribution, i.e., solved parameters are spectrally decomposed into reference (low-frequency) components estimated from a global geopotential model (GGM) and residual (high-frequency) components computed from local ground gravity and elevation data.

Reference components of selected gravity field parameters, namely geoidal undulations, deflections of the vertical and vertical gradient of anomalous gravity, are easily evaluated through the spherical harmonic synthesis. As input values, the Stokes coefficients in available GGMs can be used. Although many new solutions were published recently, the EGM96 is used in this study. Determination of the residual components from local ground gravity relies upon a solution of geodetic boundary-value problems that require numerical evaluation of surface convolutive integrals. The residual component of the geoidal undulation requires numerical evaluation of the adapted *Stokes (1849)* formula. Molodensky-modified spheroidal Stokes's formula is used in this contribution *(Vaníček and Kleusberg, 1987; see also Vaníček and Sjöberg, 1991; Martinec and Vaníček, 1996)*. The integration domain for the modified Stokes integral is divided into a spherical cap centered at the computation point, and the remainder of the sphere. The contribution of gravity data within the spherical cap is computed by discrete numerical integration. The contribution of gravity data in the region outside the spherical cap is computed by the spheroidal Molodensky-type approach *(Martinec and Vaníček, 1996)*. The residual component of the deflection of the vertical is computed by the adapted *Vening-Meinesz (1928)* formula. Its numerical solution is also obtained by discrete numerical integration. The spheroidal Vening-Meinesz formula is used in this contribution *(Featherstone et al. 2002)*. The integration domain is again truncated into a spherical cap centered at the computation point. Finally, the anomalous vertical gradient of gravity is computed by the adopted integral formula *(Heiskanen and Moritz, 1967)*. Contributions of gravity data outside this spherical cap are in the last two cases neglected due to their small numerical values.

Thus two sources of gravity data are used: 1– discrete mean values of local ground gravity from the database GOP30x30 *(Kostelecký jr., 2004)* with the spatial resolution of 30×30 arcsec, i.e., approximately 1×1 km for determination of the residual quantities, and 2– geopotential coefficients up to degree and order 120 of the EGM96 *(Lemoine et al., 1998)* for determination of the reference quantities. The elevation data represent mean values distributed over a homogeneous geographical grid with the spatial resolution of 30×30 arcsec. This geographical grid is identical with the grid of the ground gravity data.

1. Reduction of local ground gravity data

In this section, a brief description of ground gravity reduction is given. Our task is to derive residual gravity anomalies that could be used in the potential theory, i.e., they should correspond to the residual disturbing gravity potential T harmonic everywhere outside the geoid. Positions of points of interest are given in the geocentric spherical coordinate system using a triad of spherical coordinates $\{r, \theta, \lambda\}: \; 0 \leq \theta \leq \pi, \, 0 \leq \lambda < 2\pi$. A geocentric direction Ω is given by a pair of the angular coordinates $\{\theta, \lambda\}$. It is assumed that the physical surface of the Earth can be described by a two-dimensional (2-D) "star-shaped" function $r_s(\Omega) = r_g(\Omega) + H(\Omega)$ where r_g is the geocentric radius of the geoid and H is the orthometric height. Similarly, the telluroid is given as another 2-D function $r_t(\Omega) = r_e(\theta) + H(\Omega)$ where r_e is the geocentric radius of the reference ellipsoid. Although the orthometric height does not refer exactly to the geocentric direction Ω, eventual differences are neglected in the text. Moreover, the symbol Ω is skipped if the height function H as well as the other 2-D functions above are used as a parameter of another function.

Let us start with the standard decomposition of the geopotential V into the potential components corresponding to the geoid V^g, topography V^t and atmosphere V^a, i.e.,

$$T(r,\Omega) \;=\; V^g(r,\Omega) \;+\; V^t(r,\Omega) \;+\; V^a(r,\Omega) \;-\; U(r,\theta) \;= \\ \;=\; T^g(r,\Omega) \;+\; V^t(r,\Omega) \;+\; V^a(r,\Omega)\;, \tag{1}$$

where U stands for the normal (Somigliana-Pizzetti) potential. The *reduced disturbing gravity potential* T^g is harmonic above the geoid and we seek gravity anomalies associated with this potential. Applying in Eq. (1) the spherical operator of the fundamental gravimetric equation for the gravity anomaly yields

$$\Delta g(r_s,\Omega) \;=\; \left(-\,\frac{\partial}{\partial r}\;-\;\frac{2}{r}\right)\Bigg|_{r=r_s}\Big[T^g(r,\Omega)\;+\;V^t(r,\Omega)\;+\;V^a(r,\Omega)\Big]. \quad (2)$$

The ground gravity anomaly is thus decomposed into three components: 1– gravity anomaly related to the reduced disturbing gravity potential T^g harmonic outside the geoid, i.e., a *reduced ground gravity anomaly*

$$\Delta g^g(r_s,\Omega) \;=\; -\,\frac{\partial T^g(r,\Omega)}{\partial r}\Bigg|_{r=r_s}\;-\;\frac{2}{r_s}\,T^g(r_s,\Omega)\;, \quad (3)$$

2– topographical component including so-called *direct and secondary indirect topographical effects*

$$-\,\frac{\partial V^t(r,\Omega)}{\partial r}\Bigg|_{r=r_s}\;-\;\frac{2}{r_s}\,V^t(r_s,\Omega)\;=\;A^t(r_s,\Omega)\;+\;S^t(r_s,\Omega)\;, \quad (4)$$

and finally, 3– atmospheric component with *direct and secondary indirect atmospheric effects*

$$-\,\frac{\partial V^a(r,\Omega)}{\partial r}\Bigg|_{r=r_s}\;-\;\frac{2}{r_s}\,V^a(r_s,\Omega)\;=\;A^a(r_s,\Omega)\;+\;S^a(r_s,\Omega)\;. \quad (5)$$

Using the standard definition of the ground gravity anomaly as a difference of ground gravity and normal gravity at the telluroid, i.e.,

$$\Delta g(r_s,\Omega) \;=\; g(r_s,\Omega)\;-\;\gamma(r_t,\theta)\;, \quad (6)$$

one gets for evaluation of the reduced ground gravity anomaly

$$\Delta g^g(r_s,\Omega) \;=\; g(r_s,\Omega)\;-\;\gamma(r_t,\theta)\;-\;A^t(r_s,\Omega)\;-\;S^t(r_s,\Omega)\;- \\ -\;A^a(r_s,\Omega)\;-\;S^a(r_s,\Omega)\;. \quad (7)$$

Following the spectral decomposition of the gravity field, the residual gravity anomaly can be obtained through additional reduction for the reference (low-frequency) gravity anomaly synthesized from the global geopotential model (GGM) that again contains a signal generated by masses inside the geoid, topography and atmosphere, i.e.,

$$\Delta g_\ell(r_s,\Omega) \;=\; \Delta g^g_\ell(r_s,\Omega) \;+\; A^t_\ell(r_s,\Omega) \;+\; S^t_\ell(r_s,\Omega) \;+ \\ +\; A^a_\ell(r_s,\Omega) \;+\; S^a_\ell(r_s,\Omega). \tag{8}$$

The maximum degree $\ell = 120$ for the reference field will be used in all computations presented in this article. Reasoning for this selection is given in Section 3. The *residual reduced ground gravity anomaly* is then

$$\Delta g^{g,\ell}(r_s,\Omega) \;=\; \Delta g^\ell(r_s,\Omega) \;-\; A^{t,\ell}(r_s,\Omega) \;-\; S^{t,\ell}(r_s,\Omega) \;- \\ -\; A^{a,\ell}(r_s,\Omega) \;-\; S^{a,\ell}(r_s,\Omega)\;. \tag{9}$$

The disturbing gravity potential corresponding to $\Delta g^{g,\ell}$ is defined and solved in Section 3.1. Thus, only the residual direct and secondary indirect topographical and atmospheric effects are required for the reduction. As a consequence of using the residual quantities, the residual atmospheric effects $A^{a,\ell}$ and $S^{a,\ell}$ are neglected in this study. The reduced residual ground gravity anomalies can be then written

$$\Delta g^{g,\ell}(r_s,\Omega) \;\doteq\; g(r_s,\Omega) \;-\; \gamma(r_t,\theta) \;-\; \Delta g_\ell(r_s,\Omega) \;- \\ -\; A^{t,\ell}(r_s,\Omega) \;-\; S^{t,\ell}(r_s,\Omega)\;. \tag{10}$$

In the remaining part of this section, the evaluation of the individual components in Eq. (10) is discussed.

Observed values of ground gravity g were taken from the gravity database GOP30x30 with mean values given in the grid of geographical coordinates with the resolution of 30×30 arcsec. The magnitude of normal gravity at the telluroid r_t can easily be obtained by the upward continuation of normal gravity γ that is an analytical function and thus expandable into a convergent Taylor power series with the point of expansion at the reference ellipsoid r_e

$$\gamma(r_t,\theta) \;=\; \gamma(r_e,\theta) \;+\; \frac{\partial\gamma(r,\theta)}{\partial h}\bigg|_{r=r_e} H(\Omega) \;+ \\ +\; \frac{\partial^2\gamma(r,\theta)}{\partial h^2}\bigg|_{r=r_e} \frac{H^2(\Omega)}{2} \;+\; \mathcal{O}(H^3)\;. \tag{11}$$

H stands for the known topographical height corresponding to the geocentric direction Ω and h is the direction of the ellipsoidal surface normal at co-latitude θ (differences between the geocentric and ellipsoidal co-latitudes are neglected). Landau's symbol $\mathcal{O}$ represents the order of magnitude of the first neglected term in the infinite power series. Normal gravity at the reference ellipsoid can be computed using the Somigliana-Pizzetti formula (*Pizzetti, 1911*; see also *Somigliana, 1929* and *Moritz, 1984*)

$$\gamma(r_e, \theta) \;=\; \gamma_\varepsilon \, \frac{1 + k \;\cos^2\theta}{(1 - e^2 \cos^2\theta)^{\frac{1}{2}}} \;, \tag{12}$$

where γ_ε is the magnitude of normal gravity at the ellipsoidal equator, k is a numerical constant of the normal gravity formula, and e is the first numerical eccentricity of the reference ellipsoid GRS80 *(Moritz, 1984)*. It is acknowledged that at least the linear and the quadratic terms of the Taylor power series in Eq. (11) should be used to obtain satisfactory accuracy for normal gravity at the telluroid. The first-order vertical derivative of normal gravity reads *(Heiskanen and Moritz, 1967)*

$$\left.\frac{\partial\gamma(r,\theta)}{\partial h}\right|_{r=r_e} \;=\; -\,2\,\frac{\gamma(r_e,\theta)}{a}\;(\,1 \,+\, f\sin 2\theta \,+\, m\,) \,+\, \mathcal{O}(f^2)\;, \tag{13}$$

and the second-order vertical derivative

$$\left.\frac{\partial^2\gamma(r,\theta)}{\partial h^2}\right|_{r=r_e} \;=\; 6\,\frac{\gamma(r_e,\theta)}{a^2\;(1 - f\cos^2\theta)^2} \,+\, \mathcal{O}(f^2)\;. \tag{14}$$

The geodetic parameter m in Eq. (13) is defined as follows *(Grossmann, 1976)*

$$m \;\doteq\; \frac{a\,\omega^2}{\gamma_\varepsilon}\;, \tag{15}$$

where a stands for the major semi-axis of the reference ellipsoid, f for its flattening, and ω for the mean angular velocity of the Earth's rotation *(Moritz, 1984)*. Neglecting higher-order terms, the final expression for the evaluation of normal gravity at the telluroid is

$$\gamma(r_t,\theta) \;\doteq\; \gamma_\varepsilon\,\frac{1 + k\cos^2\theta}{(1 - e^2\cos^2\theta)^{\frac{1}{2}}}\left[\,1 \,-\, \frac{2\,H(\Omega)}{a}\,(1 \,+\, f\sin 2\theta \,+\, m) \,+\, \frac{3\,H^2(\Omega)}{a^2(1 - f\cos^2\theta)^2}\,\right]. \tag{16}$$

The reference gravity anomaly Δg_ℓ can be synthesized

$$\forall\ \mu\ =\ \frac{R}{r_s(\Omega)}\ \leq\ 1\ :\ \Delta g_\ell(r_s,\Omega)\ =\ \frac{GM}{R^2}\ \sum_{n=2}^{\ell}\ (n-1)\ \mu^{n+2}\ T_n(\Omega)\ , \qquad (17)$$

where GM is the product of the Newtonian gravitational constant G and the mass of the Earth M, and R is the radius of the geocentric sphere upon which the spherical harmonic expansion of the coefficients T_n reduces to the Laplace harmonics. Using the polar coordinates $\{\alpha,\psi\}$, Laplace's harmonics of degree n can be computed as follows

$$T_n(\Omega)\ =\ \sum_{m=-n}^{n}\ T_{n,m}\ Y_{n,m}(\Omega)\ = \\ =\ \frac{2n+1}{4\pi}\ \int_0^{2\pi}\int_0^{\pi} T(R,\alpha,\psi)\ P_n(\cos\psi)\ \sin\psi\ \mathrm{d}\psi\ \mathrm{d}\alpha\ . \qquad (18)$$

The corresponding spherical distance ψ is defined by the law of cosine

$$\cos\psi\ =\ \cos\theta\ \cos\theta^{'}\ +\ \sin\theta\ \sin\theta^{'}\cos\,(\lambda-\lambda^{'})\ . \qquad (19)$$

Similarly, the azimuth α can be computed by spherical trigonometry

$$\tan\alpha\ =\ \frac{\sin\theta^{'}\ \sin(\lambda^{'}-\lambda)}{\sin\theta\ \cos\theta'-\cos\theta\ \sin\theta'\ \cos(\lambda'-\lambda)}\ . \qquad (20)$$

Fully-normalized, unitless spherical harmonic coefficients $T_{n,m}$ of degree n and order m are obtained from the EGM96 coefficients reduced for their counterparts representing the normal gravity field. Spherical harmonic functions $Y_{n,m}$ read *(Hobson, 1931)*

$$\forall\ n\geq m\ :\ Y_{n,m}(\Omega)\ =\ e^{im\lambda}\ P_{n,m}(\cos\theta)\ , \qquad (21)$$

with the associated trigonometric Legendre function $P_{n,m}$ of the first kind *(Hobson, 1931)*.

The residual direct topographical effect $A^{t,\ell}$ as well as the residual secondary indirect topographical effect $S^{t,\ell}$ should be computed through their respective total values reduced for reference components. This requires demanding computations involving global topographical data. The alternative approach used in this article is based on the assumption that local topography defined as masses within a certain distance from the gravity station is responsible mainly for the residual component of the direct and secondary indirect topographical effects. They are referred to as direct and secondary indirect terrain effects. It should be emphasized that this distinction holds only approximately since local topographical masses generate a signal that covers the entire frequency spectrum. The residual direct terrain effect can approximately be computed as follows *(Novák, 2001)*:

$$\forall \nu = \frac{r_s(\Omega')}{r_s(\Omega)} : A^{t,\ell}(r_s,\Omega) \approx G\varrho\, r_s(\Omega) \int_0^{2\pi}\int_0^{\psi_o} \mathcal{K}(\nu,\psi)\, \sin\psi \; \mathrm{d}\psi \; \mathrm{d}\alpha \; , \tag{22}$$

with the mean topographical mass density ϱ and the integration kernel *(Gradshteyn and Ryzhik, 1980)*

$$\begin{aligned} \mathcal{K}(\nu,\psi) \;=\; & \frac{\nu^2 \cos\psi \;+\; \nu\,(1 \;-\; 6\cos^2\psi) \;+\; 3\cos\psi}{\mathcal{L}(\nu,\psi)} \;- \\ & - \frac{1 \;+\; 4\cos\psi \;-\; 6\cos^2\psi}{\mathcal{L}(\psi)} \;+ \\ & + (3\cos^2\psi \;-\; 1)\; \ln\left| \frac{\nu \;-\; \cos\psi \;+\; \mathcal{L}(\nu,\psi)}{1 \;-\; \cos\psi \;+\; \mathcal{L}(\psi)} \right| \; . \end{aligned} \tag{23}$$

The normalized Euclidean distance functions between the computation and integration points read

$$\mathcal{L}(\nu,\psi) \;=\; \sqrt{1 \;+\; \nu^2 \;-\; 2\,\nu\cos\psi}\;, \qquad \mathcal{L}(\psi) \;=\; 2\;\sin\frac{\psi}{2}\;. \tag{24}$$

The radius ψ_o should correspond to the maximum degree ℓ of the reference field. The secondary indirect terrain effect can similarly be computed as follows *(Novák, 2001)*:

$$\forall \nu = \frac{r_s(\Omega')}{r_s(\Omega)} : S^{t,\ell}(r_s,\Omega) \approx G\varrho\, r_s(\Omega) \int_0^{2\pi}\int_0^{\psi_o} \mathcal{N}(\nu,\psi)\, \sin\psi \; \mathrm{d}\psi \; \mathrm{d}\alpha \; , \tag{25}$$

with the integration kernel *(Gradshteyn and Ryzhik, 1980)*

$$\begin{aligned} \mathcal{N}(\nu,\psi) \;=\; & (\nu \;+\; 3\cos\psi)\; \mathcal{L}(\nu,\psi) \;-\; (1 \;+\; 3\cos\psi)\; \mathcal{L}(\psi) \;+ \\ & + (3\cos^2\psi \;-\; 1)\; \ln\left| \frac{\nu \;-\; \cos\psi \;+\; \mathcal{L}(\nu,\psi)}{1 \;-\; \cos\psi \;+\; \mathcal{L}(\psi)} \right| \; . \end{aligned} \tag{26}$$

Thus, both the direct and secondary indirect terrain effects used for reduction of the residual ground gravity anomalies can be computed using one integration combining Eqs. (22) and (25). The evaluation of all topographical effects is very important and the use of the average mass density $\varrho = 2.67$ g/cm^3 is not satisfactory. A new database completed for laterally-varying mass density is being compiled. In this regards, numerical values presented at this contribution can be seen as preliminary only. Results based on the new data will be presented in an independent study.

The final remark concerns approximations involved in the formulations used throughout this section: there is no doubt that all parameters should be computed using ellipsoidal coordinates and approximations. This concerns the reference gravity anomaly, normal gravity, direct topographical effect as well as other parameters in following sections. The ellipsoidal approach was also pursued for numerical evaluations either in a form of ellipsoidal representation directly (normal gravity, topographical effects) or in a form of appropriate corrections (spherical harmonic expansions).

3. Evaluation of selected gravity field parameters

Ground gravity data usually provide accurate information on the Earth's gravity field. However, due to an insufficient ground gravity data coverage, often restricted availability and long-wavelength biases in observed gravity data (e.g., due to the drift of gravimeters), the long-wavelength component of the Earth's gravity potential cannot be well determined from ground gravity observations. At present, the reliable global information about the Earth's gravity field is available in a form of the GGM. However, due to an exponential attenuation of the Earth's gravity field with an increasing distance from the Earth, see Eq. (17), only the low-frequency component of the Earth's gravity field can reliably be detected in this way.

In this manuscript, a spectral form of the Earth's gravity potential is used, with a distinction made between the low and high-frequency components. The threshold value of $\ell = 120$ is used throughout this manuscript. This value is selected with respect to the cap radius used in local integration. It also reflects the assumption that the frequencies up to this degree can correctly be derived from the GGM and do not have to be further improved by ground gravity data.

3.1. Geoidal undulations

Based on the above frequency decomposition of the gravity data, the geoid can similarly be decomposed into the reference geoid N_ℓ and the residual geoid N^ℓ. The reference geoid can be computed in spherical approximation *(Heiskanen and Moritz, 1967)*

$$N_\ell(\Omega) \;=\; \frac{GM}{R\,\gamma(r_e,\theta)} \sum_{n=2}^{\ell} T_n(\Omega)\;, \tag{27}$$

where γ is normal gravity at the reference ellipsoid.

The geodetic boundary-value problem is then used for the solution of the residual geoid N^ℓ. It is assumed that there are no external masses outside the geoid or, alternatively, that their gravitational effect was removed from the ground gravity data by the parameter $A^{t,\ell}$ and the reference gravity anomaly Δg_ℓ, respectively; see Eq. (10). Then the corresponding residual disturbing gravity potential $T^{g,\ell}$ is a harmonic function everywhere outside the geoid and its behaviour is controlled by the Laplace differential equation *(Heiskanen and Moritz, 1967)*

$$\forall\, r \;>\; r_g \;:\; \nabla^2\, T^{g,\ell}(r,\Omega) \;=\; 0\;. \tag{28}$$

A boundary condition to the homogeneous elliptical equation (28) for the solution of the unknown function $T^{g,\ell}$ in terms of the residual reduced ground gravity anomaly $\Delta g^{g,\ell}$ in Eq. (10) is the fundamental gravimetric equation, which reads in the spherical approximation as follows *(Martinec and Vaníček, 1996)*:

$$\forall\, r = r_s \;:\; -\,\Delta g^{g,\ell}(r,\Omega) \;\doteq\; \frac{\partial}{\partial r}\, T^{g,\ell}(r,\Omega)\,\bigg|_{r} \;+\; \frac{2}{r}\, T^{g,\ell}(r,\Omega)\;. \tag{29}$$

The solution of Eqs. (28) and (29) for the unknown function $T^{g,\ell}$ exists and is unique when $T^{g,\ell}$ is regular at infinity and when $\Delta g^{g,\ell}$ does not contain first-degree harmonics.

Before the solution for the residual reduced disturbing gravity potential $T^{g,\ell}$ at the geoid can be written, reduced residual ground gravity anomalies in Eq. (10) must be referred to the geoid. This step is the so-called downward continuation solved by the spheroidal Abel-Poisson integral *(Kellogg, 1929)*

$$\forall\ \mu\ \leq\ 1\ :\ \Delta g^{g,\ell}(r_s,\Omega)\ =\ \frac{\mu}{4\pi}\int_0^{2\pi}\int_0^{\psi_o}\Delta g^{g,\ell}(R,\alpha,\psi)\ \times \times\ \mathcal{P}^{\ell}(\mu,\psi)\ \sin\psi\ \mathrm{d}\psi\ \mathrm{d}\alpha\ , \tag{30}$$

with the spheroidal Abel-Poisson function

$$\mathcal{P}^{\ell}(\mu,\psi)\ =\ \mathcal{P}(\mu,\psi)\ -\ \sum_{n=0}^{\ell}\ (2n+1)\ \mu^{n+1}\ P_n(\cos\psi)\ , \tag{31}$$

derived from its spherical form

$$\mathcal{P}(\mu,\psi)\ =\ \frac{\mu^2\ -\ 1}{\mathcal{L}^3(\mu,\psi)}\ . \tag{32}$$

The normalized Euclidean distance function $\mathcal{L}(\mu,\psi)$ is defined in Eq. (24). Legendre polynomials P_n in Eq. (31) can be generated using the recurrence relation *(Paul, 1973)*

$$\forall\ n\ \geq\ 1\ :\ P_{n+1}(\cos\psi)\ =\ \frac{2n\ -\ 1}{n}\ \cos\psi\ P_n(\cos\psi)\ - -\ \frac{n\ -\ 1}{n}\ P_{n-1}(\cos\psi)\ , \tag{33}$$

with $P_0(\cos\psi) = 1$ and $P_1(\cos\psi) = \cos\psi$, respectively.

The solution for the residual co-geoid is given by the Stokes integral

$$N^{g,\ell}(\Omega)\ =\ \frac{R}{4\pi\gamma(r_e,\theta)}\int_0^{2\pi}\int_0^{\psi_o}\Delta g^{g,\ell}(R,\alpha,\psi)\ \mathcal{S}^{\ell}(\psi)\ \sin\psi\ \mathrm{d}\psi\ \mathrm{d}\alpha\ , \tag{34}$$

where Ω defines the geocentric position of the computation point on the geoid approximated by the geocentric sphere. Equation (34) combines the Stokes integral and the Bruns theorem. The spheroidal Stokes function $\mathcal{S}^{\ell}$ for computation of the residual co-geoid in Eq. (34) reads

$$\mathcal{S}^{\ell}(\psi)\ =\ \mathcal{S}(\psi)\ -\ \sum_{n=2}^{\ell}\ \frac{2n+1}{n-1}\ P_n(\cos\psi)\ , \tag{35}$$

with the spherical Stokes function $\mathcal{S}$ *(Stokes, 1849)*

$$\mathcal{S}(\psi)\ =\ 1\ +\ \csc\frac{\psi}{2}\ -\ 6\sin\frac{\psi}{2}\ -\ 5\cos\psi\ - -\ 3\cos\psi\ \ln\left(\ \sin\frac{\psi}{2}\ +\ \sin^2\frac{\psi}{2}\ \right)\ . \tag{36}$$

The geoid is obtained by the summation of the reference geoid N_ℓ in Eq. (27), the residual co-geoid $N^{g,\ell}$ in Eq. (34) and the *primary indirect terrain effect* $N^{t,\ell}$

$$N(\Omega) \;=\; N_\ell(\Omega) \;+\; N^{g,\ell}(\Omega) \;+\; N^{t,\ell}(\Omega) \;=\; N_\ell(\Omega) \;+\; N^{\ell}(\Omega) \; . \tag{37}$$

Following the same argumentation as in the case of the direct terrain effect $A^{t,\ell}$ and secondary indirect terrain effect $S^{t,\ell}$ above, the primary indirect terrain effect can approximately be evaluated as follows *(Novák, 2001)*:

$$\forall\; \kappa' \;=\; \frac{r_s(\Omega')}{R} \;,\; \kappa \;=\; \frac{r_s(\Omega)}{R} \;:\; N^{t,\ell}(\Omega) = \\ = \frac{R^2 G\varrho}{2\;\gamma(r_e,\theta)} \int_0^{2\pi}\!\!\int_0^{\psi_o} \mathcal{J}(\kappa',\kappa,\psi)\;\sin\psi\;\mathrm{d}\psi\;\mathrm{d}\alpha \;, \tag{38}$$

with the normalized integration kernel *(Gradshteyn and Ryzhik, 1980)*

$$\mathcal{J}(\kappa',\kappa,\psi) \;=\; (\kappa' +\; 3\cos\psi)\;\mathcal{L}(\kappa',\psi) \;-\; (\kappa \;+\; 3\cos\psi)\;\mathcal{L}(\kappa,\psi) \;+ \\ +\;(3\cos^2\psi \;-\; 1)\;\ln\left|\,\frac{\kappa' \;-\; \cos\psi \;+\; \mathcal{L}(\kappa',\psi)}{\kappa \;-\; \cos\psi \;+\; \mathcal{L}(\kappa,\psi)}\,\right| \;. \tag{39}$$

3.2. Deflections of the vertical

The reference meridian component of the deflection of the vertical at the geoid can be computed from the reference gravity disturbing potential T_ℓ

$$\xi_\ell(R,\Omega) \;=\; -\;\frac{GM}{R^2\;\gamma(r_e,\theta)}\;\sum_{n=2}^{\ell}\;\frac{\partial T_n(\Omega)}{\partial\theta} \;, \tag{40}$$

and the reference prime vertical component

$$\eta_\ell(R,\Omega) \;=\; -\;\frac{GM}{R^2\;\gamma(r_e,\theta)\;\sin\theta}\;\sum_{n=2}^{\ell}\;\frac{\partial T_n(\Omega)}{\partial\lambda} \;. \tag{41}$$

Similarly to the reference geoid in Eq. (27), these quantities are also derived from the EGM96 spherical harmonic coefficients $T_{n,m}$, see Eq. (18). The horizontal derivatives of the spherical harmonic function $Y_{n,m}$ read

$$\forall\; n \geq m \;:\; \frac{\partial Y_{n,m}(\Omega)}{\partial\theta} \;=\; e^{im\lambda}\;\frac{\partial P_{n,m}(\sin\theta)}{\partial\theta} \;, \tag{42}$$

$$\forall\ n \geq m\ :\ \frac{\partial Y_{n,m}(\Omega)}{\partial \lambda}\ =\ im\ e^{im\lambda}\ P_{n,m}(\sin\theta)\ . \tag{43}$$

For their numerical evaluation, see e.g. *(Abramowitz and Stegun, 1972).*

The residual meridian component of the deflection of the vertical can be computed by the Vening-Meinesz integral *(Heiskanen and Moritz, 1967)* from the corresponding gravity data at the geoid, see Eq. (30). The solution for the meridian component has the form of the surface integral

$$\xi^{g,\ell}(R,\Omega) = \frac{1}{4\pi\gamma(r_e,\theta)}\int_0^{2\pi}\int_0^{\psi_o} \Delta g^{g,\ell}(R,\alpha,\psi)\mathcal{M}^{\ell}(\psi)\sin\psi\,\mathrm{d}\psi\,\cos\alpha\,\mathrm{d}\alpha\,, \tag{44}$$

and similarly the prime vertical component

$$\eta^{g,\ell}(R,\Omega) = \frac{1}{4\pi\gamma(r_e,\theta)}\int_0^{2\pi}\int_0^{\psi_o} \Delta g^{g,\ell}(R,\alpha,\psi)\mathcal{M}^{\ell}(\psi)\sin\psi\,\mathrm{d}\psi\,\sin\alpha\,\mathrm{d}\alpha\,. \tag{45}$$

The spheroidal Vening-Meinesz integration kernel has the form

$$\mathcal{M}^{\ell}(\psi)\ =\ \mathcal{M}(\psi)\ +\ \sum_{n=2}^{\ell}\frac{2n+1}{n-1}\ P_{n,1}(\cos\psi)\ , \tag{46}$$

where $\mathcal{M}$ is the spherical Vening-Meinesz kernel *(Heiskanen and Moritz, 1967)*

$$\mathcal{M}(\psi)\ =\ -\frac{\cos\frac{\psi}{2}}{2\sin^2\frac{\psi}{2}}\ +\ 8\sin\psi\ -\ 6\cos\frac{\psi}{2}\ -\ 3\ \frac{1-\sin\frac{\psi}{2}}{\sin\psi}\ + \\ +\ 3\sin\psi\ \ln\left[\ \sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\ \right]\ . \tag{47}$$

The associated Legendre functions $P_{n,1}$ of the order one, that represent the derivative of the Legendre polynomials, can easily be generated using another convenient recurrence relation

$$\forall\ n\geq 3\ :\ (n-1)\ P_{n,1}(\cos\psi)\ =\ \cos\psi\ (2n-1)\ P_{n-1,1}(\cos\psi)\ - \\ -\ n\ P_{n-2,1}(\cos\psi)\ , \tag{48}$$

with $P_{1,1}(\cos\psi) = -\sin\psi$ and $P_{2,1}(\cos\psi) = -3\cos\psi\ \sin\psi$. The indirect topographical effects on the deflection of the vertical are, see Eq. (38),

$$\xi^{t,\ell}(R,\Omega)\ =\ -\ \frac{1}{R}\ \frac{\partial N^{t,\ell}(\Omega)}{\partial\theta}\ , \tag{49}$$

$$\eta^{t,\ell}(R,\Omega) \;=\; -\,\frac{1}{R\sin\theta}\,\frac{\partial N^{t,\ell}(\Omega)}{\partial\lambda}\;. \tag{50}$$

The total value of the meridian component of the deflection of the vertical is then

$$\begin{aligned}\xi(R,\Omega) \;&=\; \xi_\ell(R,\Omega) \;+\; \xi^{g,\ell}(R,\Omega) \;+\; \xi^{t,\ell}(R,\Omega) = \\ &=\; \xi_\ell(R,\Omega) \;+\; \xi^{\ell}(R,\Omega)\;,\end{aligned} \tag{51}$$

and the prime vertical component

$$\begin{aligned}\eta(R,\Omega) \;&=\; \eta_\ell(R,\Omega) \;+\; \eta^{g,\ell}(R,\Omega) \;+\; \eta^{t,\ell}(R,\Omega) = \\ &=\; \eta_\ell(R,\Omega) \;+\; \eta^{\ell}(R,\Omega)\;.\end{aligned} \tag{52}$$

3.3. Vertical gradients of anomalous gravity

The vertical gradient of the gravity anomaly at the geoid in spherical approximation can be evaluated by the formula

$$\Delta g_r(R,\Omega) \;=\; \left.\frac{\partial\Delta g(r,\Omega)}{\partial r}\right|_{r=R}\;, \tag{53}$$

that will be again divided into two components: 1– vertical gradient of the reference gravity anomaly

$$\Delta g_{r,\ell}(R,\Omega) \;=\; -\,\frac{GM}{R^3}\,\sum_{n=2}^{\ell}\,(n-1)\;(n+2)\;T(\Omega)\;, \tag{54}$$

and, 2– vertical gradient of the residual gravity anomaly at the geoid evaluated by the integral formula *(Heiskanen and Moritz, 1967)*

$$\begin{aligned}\Delta g_r^{\ell}(R,\Omega) = \;&\frac{1}{2\pi R}\int_0^{2\pi}\!\!\int_0^{\psi_o}\Big[\,\Delta g^{\ell}(R,\alpha,\psi) \;- \\ &-\;\Delta g^{\ell}(R,\Omega)\,\Big]\,\mathcal{K}^{\ell}(\psi)\;\sin\psi\;\mathrm{d}\psi\;\mathrm{d}\alpha\;-\;\frac{2}{R}\,\Delta g^{\ell}(R,\Omega)\;.\end{aligned} \tag{55}$$

The spheroidal integration kernel is given by the formula

$$\mathcal{K}^{\ell}(\psi) \;=\; \mathcal{K}(\psi)\;-\;\sum_{n=2}^{\ell}\;P_n^3(\cos\psi)\;, \tag{56}$$

with the spherical kernel

$$\mathcal{K}(\psi) \;=\; \frac{1}{8\sin^3\frac{\psi}{2}} \; . \tag{57}$$

Note that in this case only the reference and residual gravity anomalies are considered.

4. Practical considerations

In this section, some practical considerations for numerical evaluation of the various surface integrals are discussed. Namely, the modification of the integral functions, evaluation of the cap integration for given discrete values of mean gravity and the far-zone effects are reviewed in the following subsections.

4.1. Geoidal undulations

Due to the incomplete coverage or availability of ground gravity data, the integral in Eq. (34) is evaluated over a limited integration domain represented by a spherical cap of radius ψ_o. The influence of the gravity information from the remainder of the globe is accounted for by using the truncation errors

$$N^{g,\ell}_{fz}(\Omega) \;=\; \frac{R}{4\pi\gamma(r_e,\theta)} \int_0^{2\pi}\!\!\int_{\psi_\mathrm{o}}^{\pi} \Delta g^{g,\ell}(R,\alpha,\psi)\, \mathcal{S}^{\ell}(\psi)\; \sin\psi \;\mathrm{d}\psi\; \mathrm{d}\alpha \;, \tag{58}$$

that can be evaluated from the GGM. To minimize this far-zone contribution, the modified spheroidal Stokes function is often used *(Vaníček and Kleusberg, 1987)*

$$\mathcal{S}^{\ell}(\psi,\psi_\mathrm{o}) \;=\; \mathcal{S}(\psi) \;-\; \sum_{n=2}^{\ell} \frac{2n+1}{n-1}\, P_n(\cos\psi) \;- \\ -\; \sum_{n=0}^{\ell} \frac{2n+1}{2}\, t_n(\psi_\mathrm{o})\; P_n(\cos\psi) \;, \tag{59}$$

where t_n are the modification coefficients. It can be shown that in order to minimize the effect of gravity from the far zones, see Eq. (58), the following integral must be minimized

$$\forall\ t_n \in \mathcal{R}^n\ :\ \min_{t_n} \left\{ \int_{\psi_o}^{\pi} \left[\ S^{\ell}(\psi, \psi_o)\ \right]^2 \sin\psi\ \mathrm{d}\psi \right\}. \tag{60}$$

The solution leads to the system of linear equations for the unknown modification coefficients t_n

$$\sum_{n=0}^{\ell} \frac{2n+1}{2}\ R_{n,m}(\psi_o)\ t_n(\psi_o)\ =\ Q_m^{\ell}(\psi_o)\ , \tag{61}$$

where the coefficients $R_{n,m}$ are *(Paul, 1973)*

$$R_{n,m}(\psi_o)\ =\ \int_{\psi_o}^{\pi} P_n(\cos\psi)\ P_m(\cos\psi)\ \sin\psi\ \mathrm{d}\psi\ . \tag{62}$$

The spheroidal truncation coefficients in Eq. (61) can then be computed as *(Molodensky et al., 1960;* see also *Heiskanen and Moritz, 1967; Martinec, 1993)*

$$Q_m^{\ell}(\psi_o)\ =\ \int_{\psi_o}^{\pi} \mathcal{S}^{\ell}(\psi)\ P_m(\cos\psi)\ \sin\psi\ \mathrm{d}\psi\ =$$

$$=\ Q_m(\psi_o)\ -\ \sum_{n=2}^{\ell} \frac{2n+1}{n-1}\ \ R_{n,m}(\psi_o)\ \ . \tag{63}$$

The contribution of the spherical cap can be evaluated by discrete summation over mean values of gravity anomalies on a regular geographic grid within the cap. After accounting for the contribution of the computation point (singularity)

$$N_{ep}^{g,\ell}(\Omega)\ =\ \frac{R}{2\gamma(r_e,\theta)}\ \Delta g^{g,\ell}(R,\Omega)\ \int_0^{\psi_o} \mathcal{S}^{\ell}(\psi,\psi_o)\ \sin\psi\ \mathrm{d}\psi\ , \tag{64}$$

the contribution of the near zone reads

$$N_{nz}^{g,\ell}(\Omega)\ =\ \frac{R}{4\pi\gamma(r_e,\theta)} \int_0^{2\pi}\!\!\int_0^{\psi_o} \left[\ \Delta g^{g,\ell}(R,\alpha,\psi)\ -\ \Delta g^{g,\ell}(R,\Omega)\ \right]\ \times$$
$$\times\ \mathcal{S}^{\ell}(\psi,\psi_o)\ \sin\psi\ \mathrm{d}\psi\ \mathrm{d}\alpha\ . \tag{65}$$

Using the mean-value theorem, the integration in Eq. (65) can be replaced by the summation over $(j-1)$ cells within the spherical cap of the product of mean values of high-frequency gravity anomalies *(Vaníček and Krakiwsky, 1986)*

$$\overline{\Delta g^{g,\ell}}(R,\Omega_k) \;=\; \frac{1}{\Delta\Omega_k} \iint_{\Delta\Omega_k} \Delta g^{g,\ell}(R,\Omega') \,\mathrm{d}\Omega' \;, \tag{66}$$

with corresponding point values of the modified spheroidal Stokes function. $\Delta\Omega_k$ is the surface area of the k-th cell. Eq. (65) then reads

$$N^{g,\ell}_{nz}(\Omega) \;=\; \frac{R}{4\pi\gamma(r_e,\theta)} \sum_{k}^{j-1} \left\{ \iint_{\Delta\Omega_k} \left[\Delta g^{g,\ell}(R,\Omega') \;-\; \Delta g^{g,\ell}(R,\Omega) \right] \times \right.$$
$$\left. \times\; \mathcal{S}^{\ell}(\psi,\psi_{\mathrm{o}}) \,\mathrm{d}\Omega' \right\} \;, \tag{67}$$

that can approximately be evaluated by quadrature of the integral

$$N^{g,\ell}_{nz}(\Omega) \;\doteq\; \frac{R}{4\pi\gamma(r_e,\theta)} \sum_{k}^{j-1} \left[\overline{\Delta g^{g,\ell}}(R,\Omega_k) \;-\; \overline{\Delta g^{g,\ell}}(R,\Omega) \right] \times$$
$$\times\; \mathcal{S}^{\ell}(\psi_k,\psi_{\mathrm{o}}) \;\Delta\Omega_k \;, \tag{68}$$

where the value of $\mathcal{S}^{\ell}(\psi_k,\psi_{\mathrm{o}})$ is evaluated for the spherical distance ψ_k between the integration point and the centre of the k-th cell.

The contribution of the far-zone gravity in Eq. (58) can be evaluated using the so-called Molodensky coefficients (weights) to account for the influence of the far zones omitted from the truncated integration in Eq. (65). These coefficients for the modified spheroidal Stokes function are *(Martinec, 1993)*

$$\tilde{Q}^{\ell}_m(\psi_{\mathrm{o}}) \;=\; \int_{\psi_{\mathrm{o}}}^{\pi} \mathcal{S}^{\ell}(\psi,\psi_{\mathrm{o}}) \; P_m(\cos\psi) \; \sin\psi \;\mathrm{d}\psi \;=$$
$$=\; Q_m(\psi_{\mathrm{o}}) - \sum_{n=2}^{\ell} \frac{2n+1}{n-1} \; R_{n,m}(\psi_{\mathrm{o}}) \;-$$
$$-\sum_{n=0}^{\ell} \frac{2n+1}{2} \; t_n(\psi_{\mathrm{o}}) \; R_{n,m}(\psi_{\mathrm{o}}) \;, \tag{69}$$

where the spherical truncation coefficients $Q_m(\psi_o)$ are given by

$$Q_m(\psi_o) \; = \; \int_{\psi_o}^{\pi} \mathcal{S}(\psi) \; P_m(\cos\psi) \; \sin\psi \; \mathrm{d}\psi \; . \tag{70}$$

The contribution of the far zone to the residual geoid can then be evaluated using a conversion of the spatial form in Eq. (58) to a spectral form based on the spherical harmonic expansion

$$N_{fz}^{g,\ell}(\Omega) \; = \; \frac{R}{2} \sum_{n=\ell+1}^{max} (n-1) \; \tilde{Q}_n^{\ell}(\psi_o) \; T_n(\Omega) \; , \tag{71}$$

with the spherical harmonic coefficients of the disturbing gravity potential T_n up to degree and order $max = 360$ taken from the EGM96.

The final expression for the determination of the residual co-geoid can be written as follows:

$$\begin{aligned} N^{g,\ell}(\Omega) \; &= \; N_{ep}^{g,\ell}(\Omega) \; + \; N_{nz}^{g,\ell}(\Omega) \; + \; N_{fz}^{g,\ell}(\Omega) \doteq \\ &\doteq \; \frac{R}{2\gamma(r_e,\theta)} \; \Delta g^{g,\ell}(R,\Omega) \; \int_0^{\psi_o} \mathcal{S}^{\ell}(\psi,\psi_o) \; \sin\psi \; \mathrm{d}\psi \; + \\ &\quad + \; \frac{R}{4\pi\gamma(r_e,\theta)} \; \sum_{k}^{j-1} \; \Big[\; \overline{\Delta g^{g,\ell}}(R,\Omega_k) - \overline{\Delta g^{g,\ell}}(R,\Omega) \; \Big] \; \times \\ &\quad \times \; \mathcal{S}^{\ell}(\psi_k,\psi_o) \; \Delta\Omega_k + \; \frac{R}{2} \sum_{n=\ell+1}^{max} (n-1) \; \tilde{Q}_n^{\ell}(\psi_o) \; T_n(\Omega) \; . \end{aligned} \tag{72}$$

4.2. Deflections of the vertical

Due to the limited data area, the residual meridian component of the deflection of the vertical was computed by the truncated Vening-Meinesz integral

$$\xi^{g,\ell}(R,\Omega) = \frac{1}{4\pi\gamma(r_e,\theta)} \int_0^{2\pi}\int_0^{\psi_o} \Delta g^{g,\ell}(R,\alpha,\psi)\mathcal{M}^{\ell}(\psi)\sin\psi\,\mathrm{d}\psi\,\cos\alpha\,\mathrm{d}\alpha\,, \tag{73}$$

and similarly the prime vertical component

$$\eta^{g,\ell}(R,\Omega) = \frac{1}{4\pi\gamma(r_e,\theta)} \int_0^{2\pi}\int_0^{\psi_o} \Delta g^{g,\ell}(R,\alpha,\psi)\mathcal{M}^{\ell}(\psi)\sin\psi\,\mathrm{d}\psi\,\sin\alpha\,\mathrm{d}\alpha\,. \tag{74}$$

Discretizing the integrals yields

$$\xi^{g,\ell}_{nz}(R,\Omega) \; = \; \frac{1}{4\pi\gamma(r_e,\theta)} \sum_{k}^{j-1} \overline{\Delta g^{g,\ell}}(R,\Omega_k)\; \mathcal{M}^{\ell}(\psi_k)\; \cos\alpha_k \; \Delta\Omega_k \; , \tag{75}$$

for the residual meridian component, and

$$\eta^{g,\ell}_{nz}(R,\Omega) \; = \; \frac{1}{4\pi\gamma(r_e,\theta)} \sum_{k}^{j-1} \overline{\Delta g^{g,\ell}}(R,\Omega_k)\; \mathcal{M}^{\ell}(\psi_k)\; \sin\alpha_k \; \Delta\Omega_k \; , \tag{76}$$

for the residual prime vertical component of the deflection of the vertical, respectively. The contribution of the computation points was computed *(Heiskanen and Moritz, 1967)*

$$\xi^{g,\ell}_{ep}(R,\Omega) \; = \; - \, \frac{\psi_o}{2\gamma(r_e,\theta)} \; \frac{\partial \Delta g^{g,\ell}(R,\Omega)}{\partial\theta} \Bigg|_{\theta} \; , \tag{77}$$

for the residual meridian component, and

$$\eta^{g,\ell}_{ep}(R,\Omega) \; = \; - \, \frac{\psi_o}{2\gamma(r_e,\theta)} \; \frac{\partial \Delta g^{g,\ell}(R,\Omega)}{\partial\lambda} \Bigg|_{\lambda} \; , \tag{78}$$

for the residual prime vertical component, respectively. Horizontal derivatives of the gravity anomalies in the principle directions were computed by fitting higher-order polynomials into the data distributed along these particular directions. Taking derivatives of these polynomials, the corresponding values were evaluated numerically. Due to the fast attenuation of the integral with the spherical distance, the integration kernel was not modified and the effect of the far-zone gravity data was neglected, i.e.,

$$\xi^{\ell}_{fz}(R,\Omega) \; = \; \eta^{\ell}_{fz}(R,\Omega) \; = \; 0 \; . \tag{79}$$

4.3. Vertical gradients of anomalous gravity

The vertical gradient of the residual gravity anomaly, see Eq. (55), was evaluated by the discretized formula

$$\frac{\partial \Delta g^{\ell}_{nz}(r,\Omega)}{\partial r}\bigg|_{r=R} = \frac{1}{2\pi R}\sum_{k}^{j-1}\Big[\,\overline{\Delta g^{\ell}}(R,\Omega_k) - \overline{\Delta g^{\ell}}(R,\Omega)\,\Big]\,\mathcal{K}^{\ell}(\psi_k)\;\Delta\Omega_k - \\ - \frac{2}{R}\,\Delta g^{\ell}(R,\Omega)\;. \tag{80}$$

The contribution of the singularity to the vertical gradient of residual anomalous gravity was computed by the formula *(Heiskanen and Moritz, 1967)*

$$\frac{\partial \Delta g^{\ell}_{ep}(r,\Omega)}{\partial r}\bigg|_{r=R} = \frac{\psi_{\mathrm{o}}}{4}\left[\;\frac{\partial^2 \Delta g^{\ell}(R,\Omega)}{\partial \theta^2}\bigg|_{\theta} + \frac{\partial^2 \Delta g^{\ell}(R,\Omega)}{\partial \lambda^2}\bigg|_{\lambda}\;\right]. \tag{81}$$

The second-order horizontal derivatives of the gravity anomalies in the principle directions were computed by fitting higher-order polynomials into the data distributed along these particular directions. Taking the second-order derivatives of these polynomials, the corresponding values were evaluated numerically. Due to the fast attenuation of the integral with the spherical distance, the integration kernel was not modified and the effect of far-zone gravity was again neglected, i.e.,

$$\frac{\partial \Delta g^{\ell}_{fz}(r,\Omega)}{\partial r}\bigg|_{r=R} = 0\;. \tag{82}$$

5. Numerical results

Input elevation and gravity data from the database GOP30x30 are described in *(Kostelecký jr., 2004)*. The database includes high resolution and accuracy gravity and elevation data for the area of Central Europe. The data were compiled from individual sources and checked for their consistency and accuracy in overlapping areas. Although original data entering the database had a varying resolution and form, the database represents one of the best sources of available local gravity data for the region. Due to the different resolution and form of gravity data, it is impossible to characterize their accuracy with simple statistical estimates valid for the entire data area. Generally, gravity data have the best quality for the territory of the former Czechoslovak Republic and its neighboring areas. In this particular case, discrete observations of gravity could be used for computation of mean gravity values. The situation is then relatively better concerning elevation data where the global topographical model GTOPO30 (US Geological Survey) together with national terrain models could be used. Comparing the local terrain models with the GTOPO, standard deviations based on their discrete differences at the level of tens of metres were detected *(Kostelecký jr., 2004)*. Reduced residual gravity data were evaluated according to Eq. (10) at the geographical grid with the spatial resolution of 30×30 arcsec. The data area selected according to the computational area spans between parallels of 42 and 58 arcdeg northern latitude, and meridians of 0 and 30 arcdeg eastern longitude that corresponds to 6,912,000 data points.

The smaller computation area then covers the territory of Central Europe with the boundaries at 45 and 55 arcdeg northern latitude, and 6 and 26 arcdeg eastern longitude, respectively. The computation area is smaller due to various surface integration steps when results are evaluated over a smaller area to avoid the so-called edge effects in results of the integral formulas. The computation area corresponds to 2,880,000 points at the resolution of 30×30 arcsec. Numerical results are presented in a form of contour plots. Due to the best gravity data for the territory of the Czech Republic and its surrounding areas, the plots cover only the area bounded by 48 and 52 arcdeg northern latitude, and 11 and 20 arcdeg eastern longitude, respectively. This area corresponds to 518,400 computation points.

The reference geoid N_ℓ over this area is shown in Fig. 1 and the residual geoid N^ℓ in Fig. 2. The geoid represents one particular equipotential surface of the Earth's gravity field approximating the mean sea level globally. As such, it may directly be linked to the gravitational potential with relatively smooth properties compared to the observed gravity signal. It may not be the best choice for geophysical investigations concerning mass anomalies in the upper mantle. However, the geoid is the reference surface (vertical datum) in one system of geodetic heights with an ultimate physical meaning – orthometric height as a distance between the geoid (mean sea level) and the surface of the Earth measured along the plumbline.

The reference meridian component of the deflection of the vertical ξ_ℓ is plotted in Fig. 3 and its residual component ξ^ℓ in Fig. 4. Corresponding values for the prime vertical component of the deflection of the vertical η are plotted in Fig. 5 and Fig. 6, respectively. Deflections of the vertical evaluated at the mean sea level represent the slope of the geoid with respect to a surface parallel with the reference ellipsoid along the local meridian and prime vertical: both values represent angles between the actual plumbline and ellipsoidal surface normal in the respective directions.

Finally, the vertical gradient of the reference gravity anomaly $\Delta g_{r,\ell}$ is shown in Fig. 7 and its residual component Δg_r^ℓ in Fig. 8. This figure is included only to complete formally figures of other computed parameters despite its high complexity. The unit of the gravity gradient is 1 Eötvös (E) $= 10^{-9}$ s^{-2} = 0.1 mGal/km. The differential quantity Δg_r is much more sensitive to smaller features of the gravity field than the above parameters. Obviously, its correct evaluation is not possible without a complete knowledge of the mass density distribution within topographical masses. Thus these values must be interpreted with care keeping this deficiency in mind. However, the same remark applies to all parameters presented in this manuscript. Elementary statistics of the results for the computation area shown at the figures, including total values of the computed parameters, are then in Table 1.

Table 1. Statistical values of the results (plotted computation area)

function	symbol	minimum	maximum	mean	sigma	rms	units
reference geoid	N_ℓ	32.602	47.913	43.271	3.209	43.391	m
residual geoid	N^ℓ	−1.451	3.451	1.144	0.839	1.419	
geoid	N	31.754	49.067	44.416	3.533	44.556	
reference deflection	ξ_ℓ	−3.774	9.413	3.290	3.123	4.536	arcsec
residual deflection	ξ^ℓ	−12.832	16.386	0.112	3.195	3.197	
deflection	ξ	−13.739	19.605	3.402	5.385	6.369	
reference deflection	η_ℓ	−1.264	4.822	2.324	1.401	2.714	arcsec
residual deflection	η^ℓ	−14.868	9.380	−0.149	2.887	2.891	
deflection	η	−13.038	12.117	2.175	3.608	4.213	
reference gradient	$\Delta g_{\ell,r}$	−3.732	3.175	0.507	1.375	1.466	E
residual gradient	Δg_r^ℓ	−293.156	161.523	0.032	25.169	25.169	
gradient	Δg_r	−291.992	162.413	0.541	25.223	25.227	

6. Conclusions

Evaluation of selected parameters of the Earth's gravity field using the high resolution ground gravity and elevation data was discussed in this contribution. Values of the geoidal undulations, deflections of the vertical, and

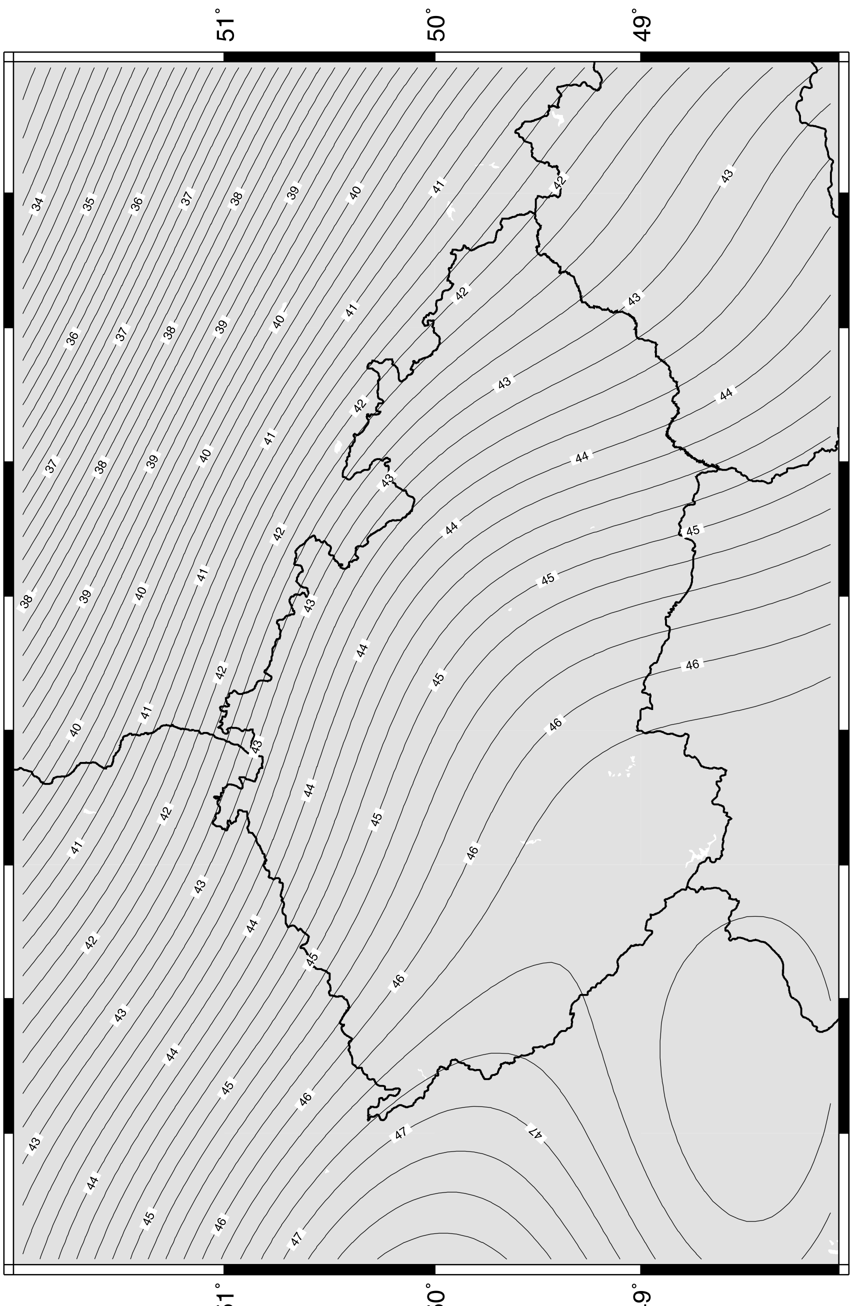

Fig. 1. Reference geoidal undulations N_ℓ (m).

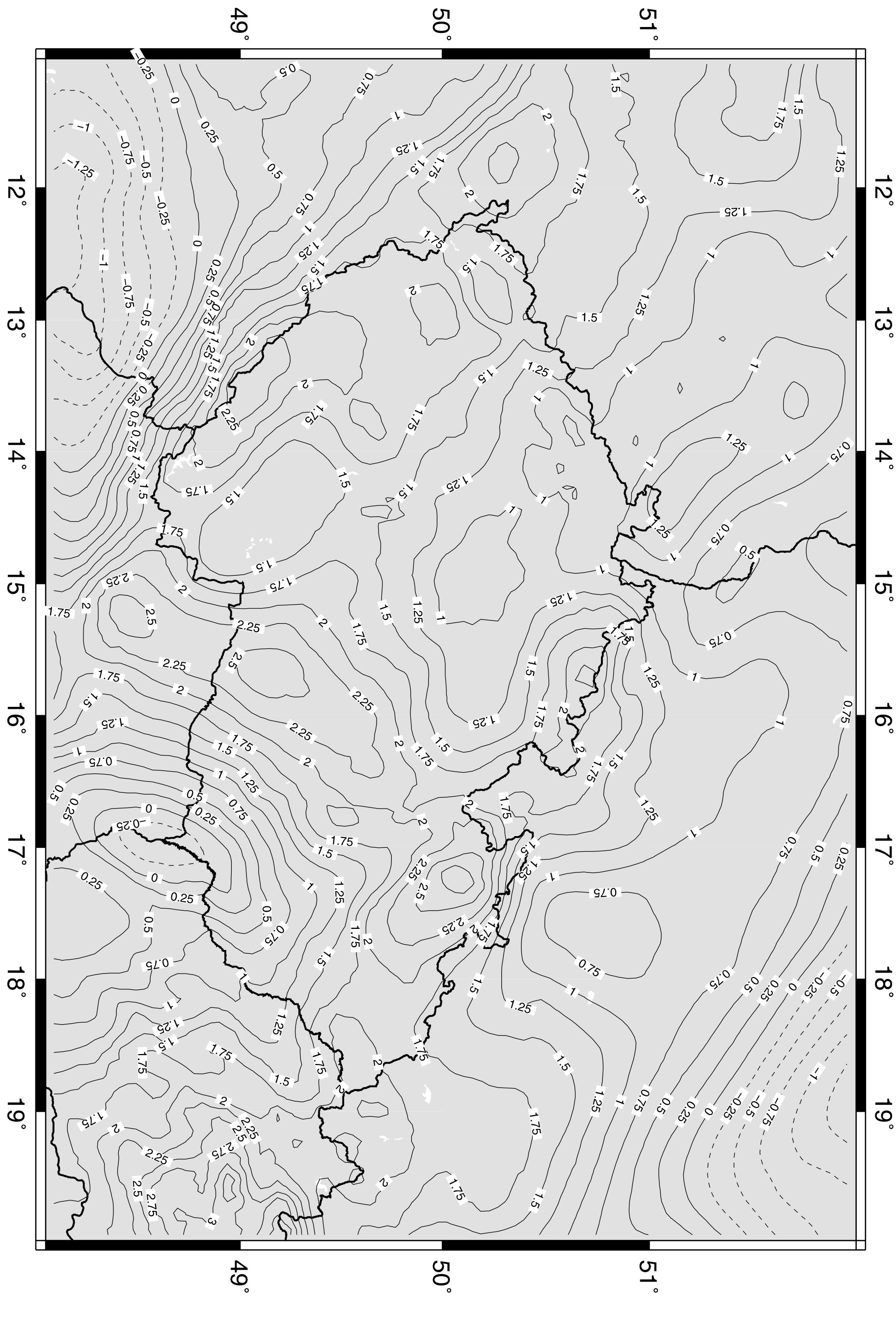

Fig. 2. Residual geoidal undulations N^{ℓ} (m).

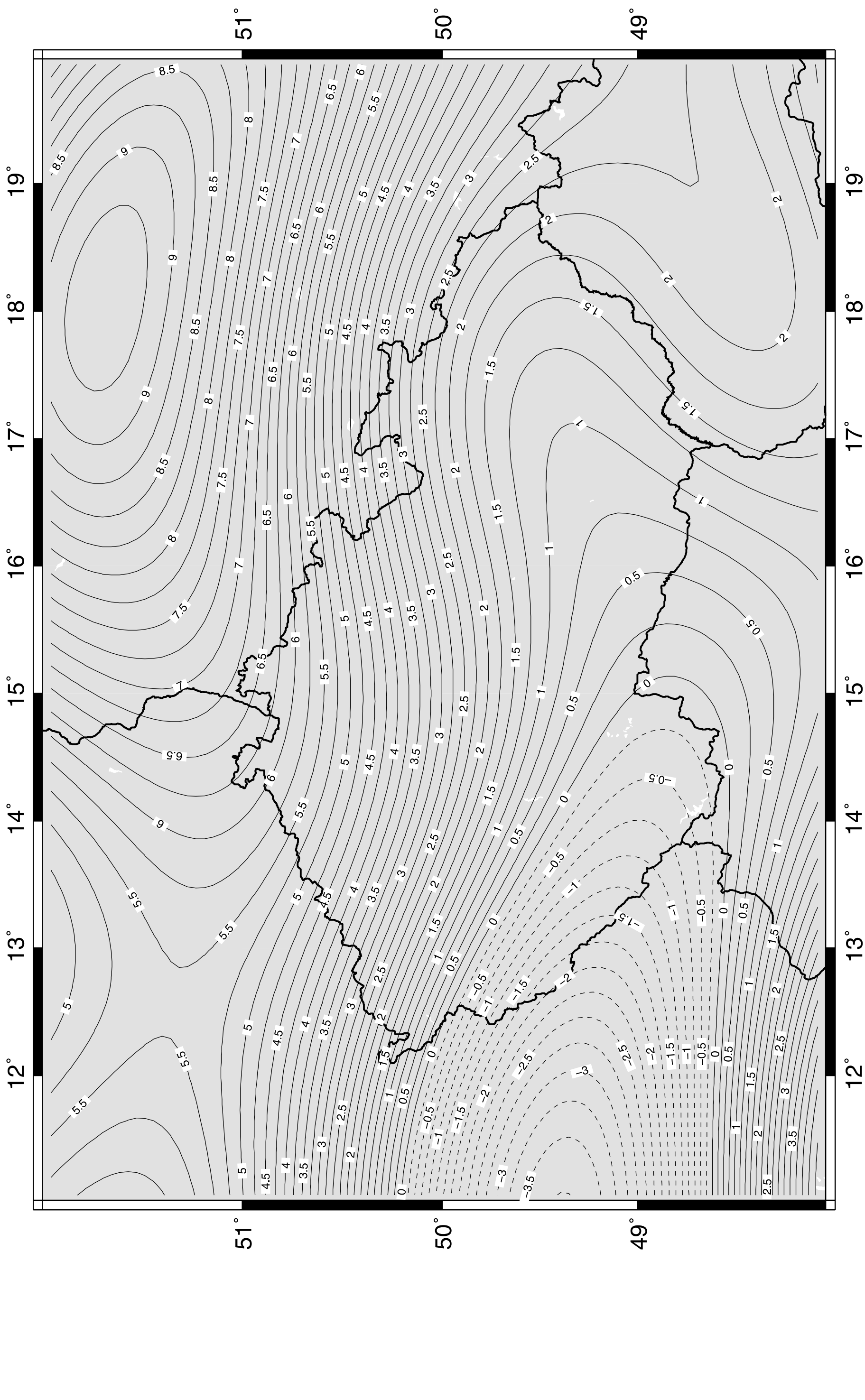

Fig. 3. Reference meridian component of the deflection of the vertical ξ_ℓ (arcsec).

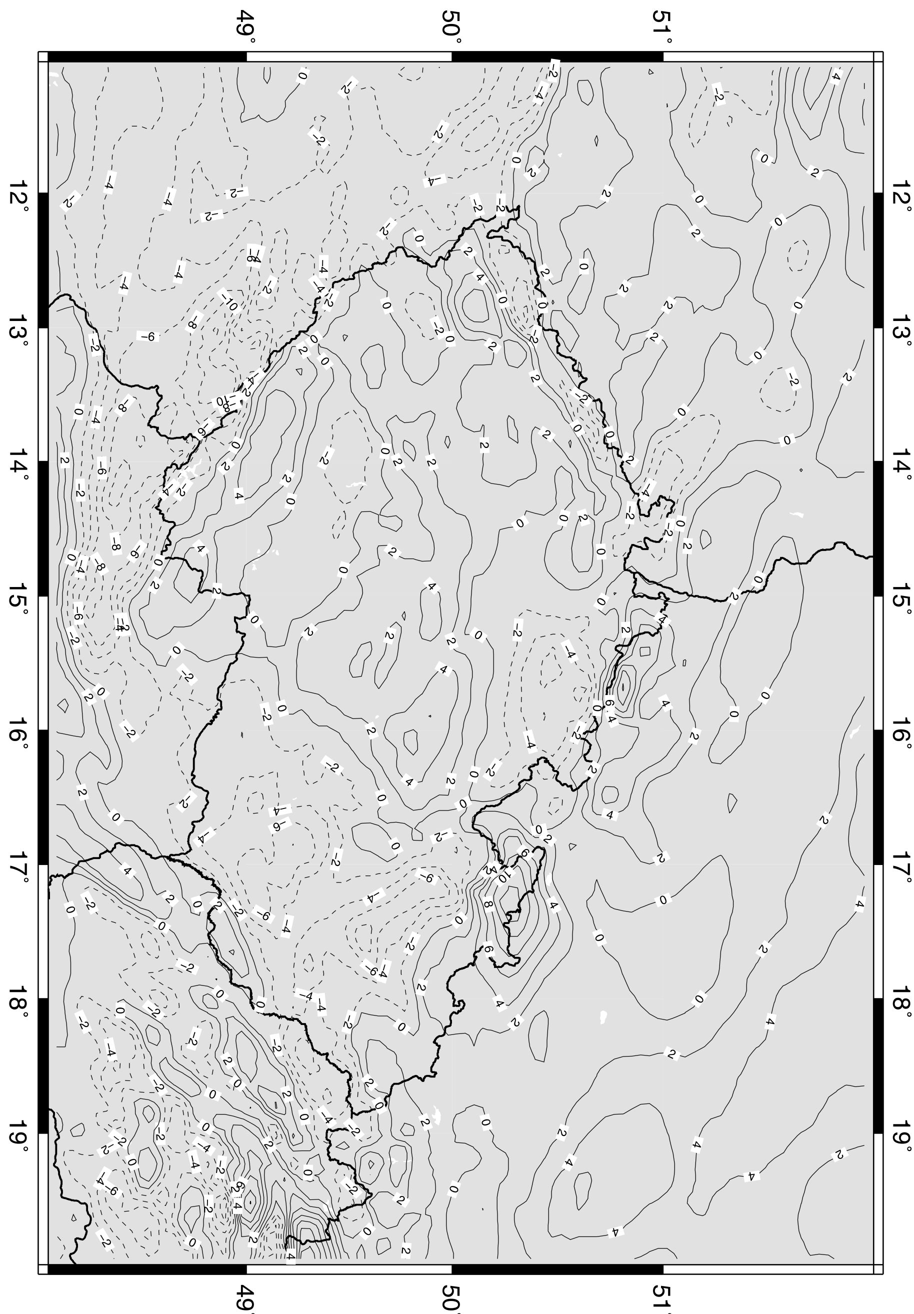

Fig. 4. Residual meridian component of the deflection of the vertical ξ^{ℓ} (arcsec).

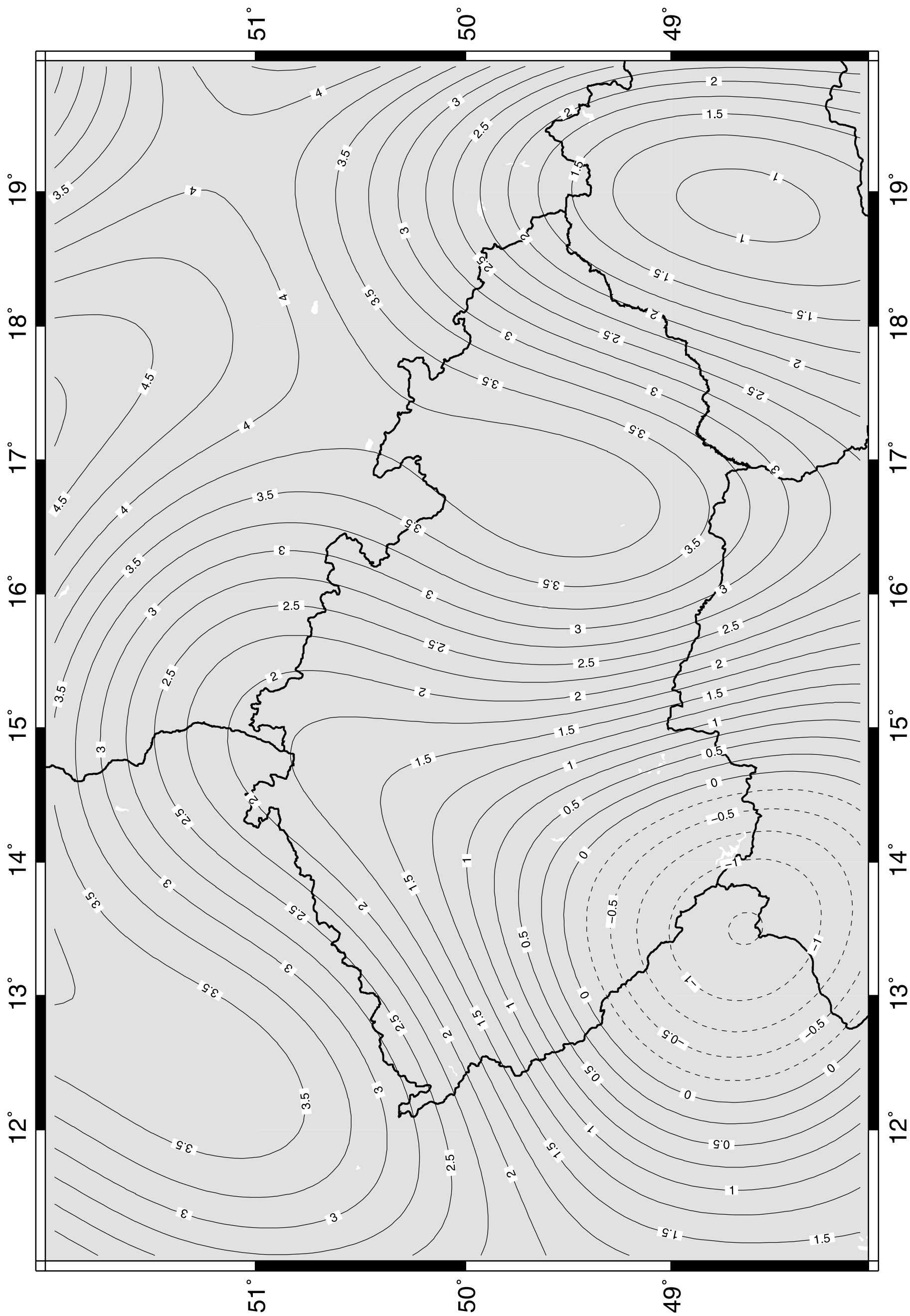

Fig. 5. Reference prime vertical component of the deflection of the vertical η_ℓ (arcsec).

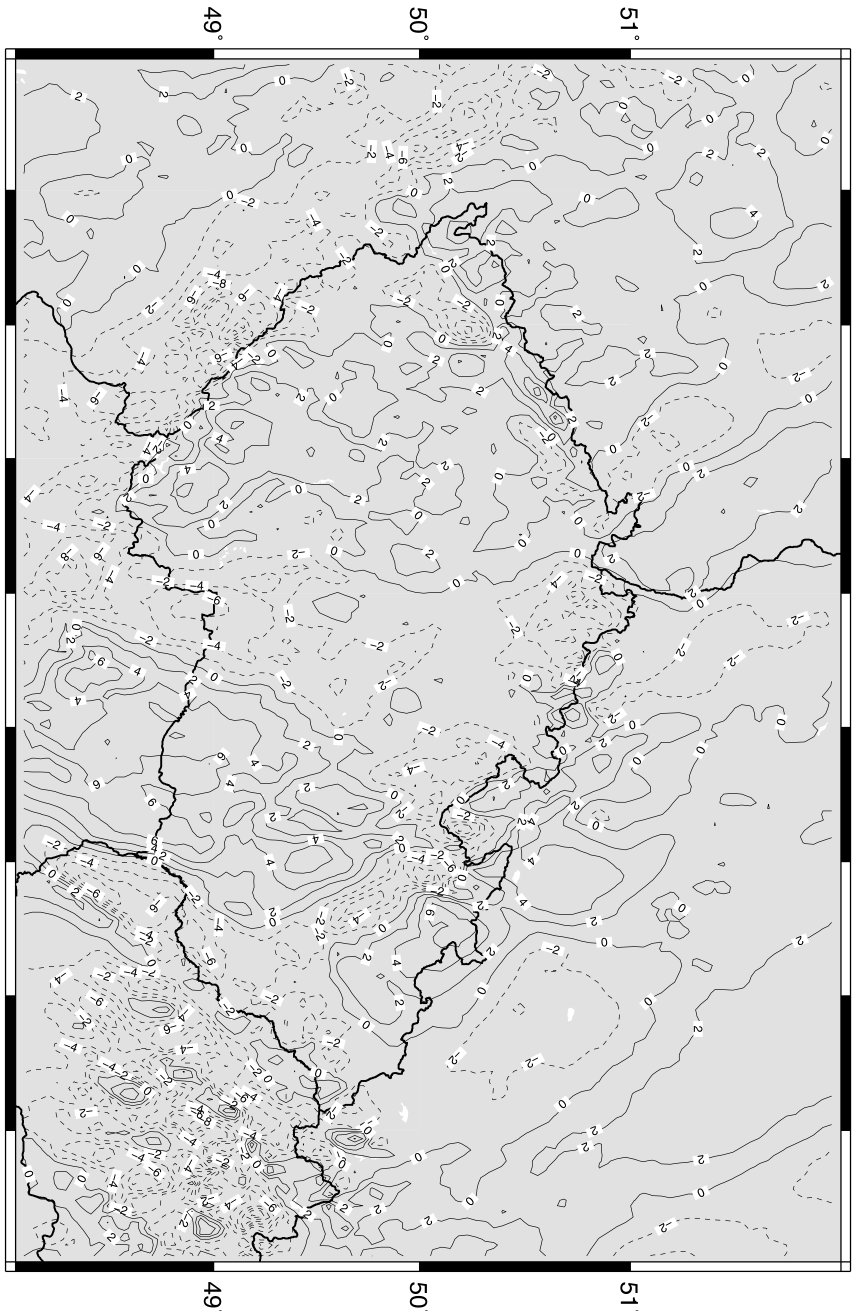

Fig. 6. Residual prime vertical component of the deflection of the vertical η^ℓ (arcsec).

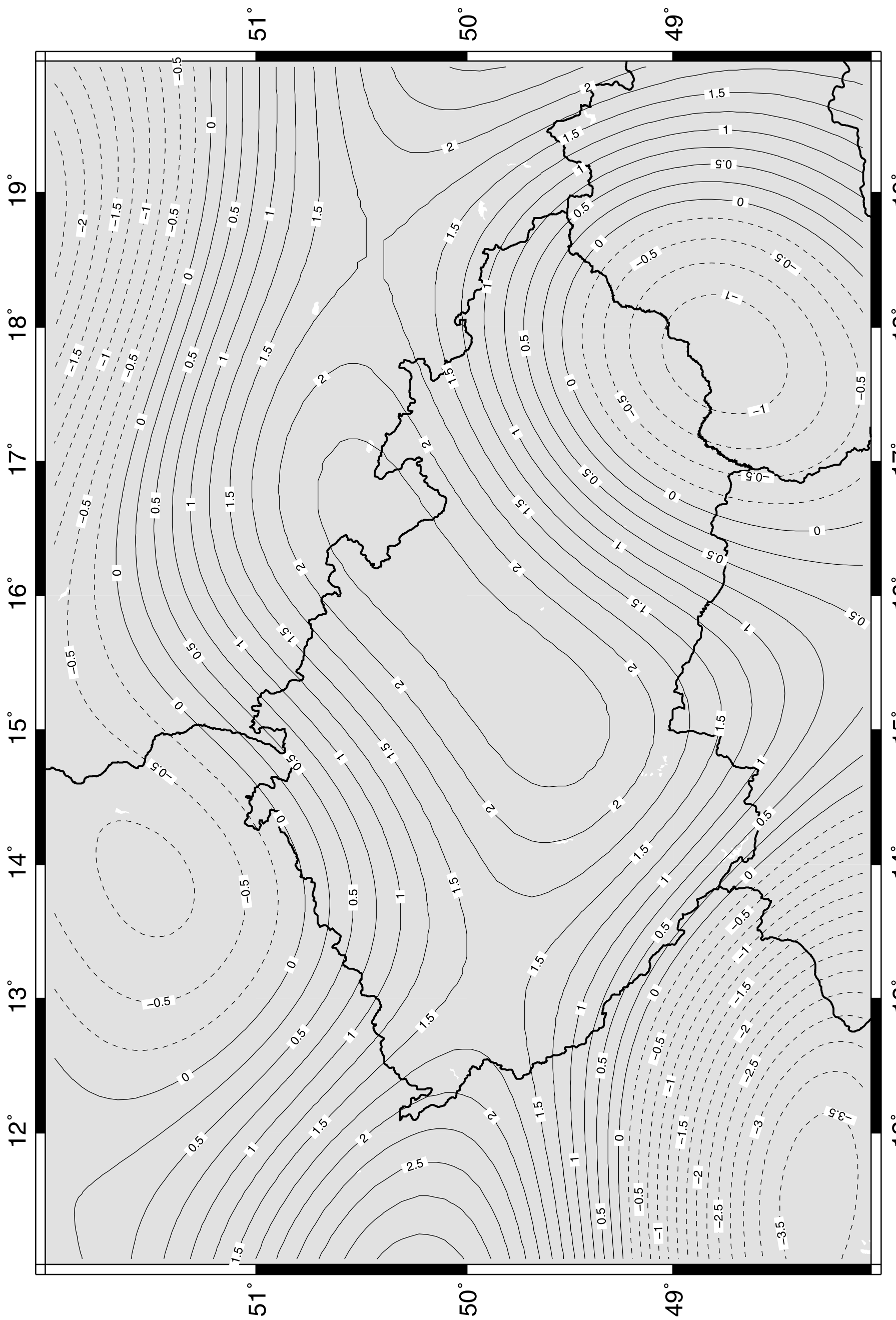

Fig. 7. Vertical gradient of the reference gravity anomaly $\Delta g_{r,\ell}$ (E).

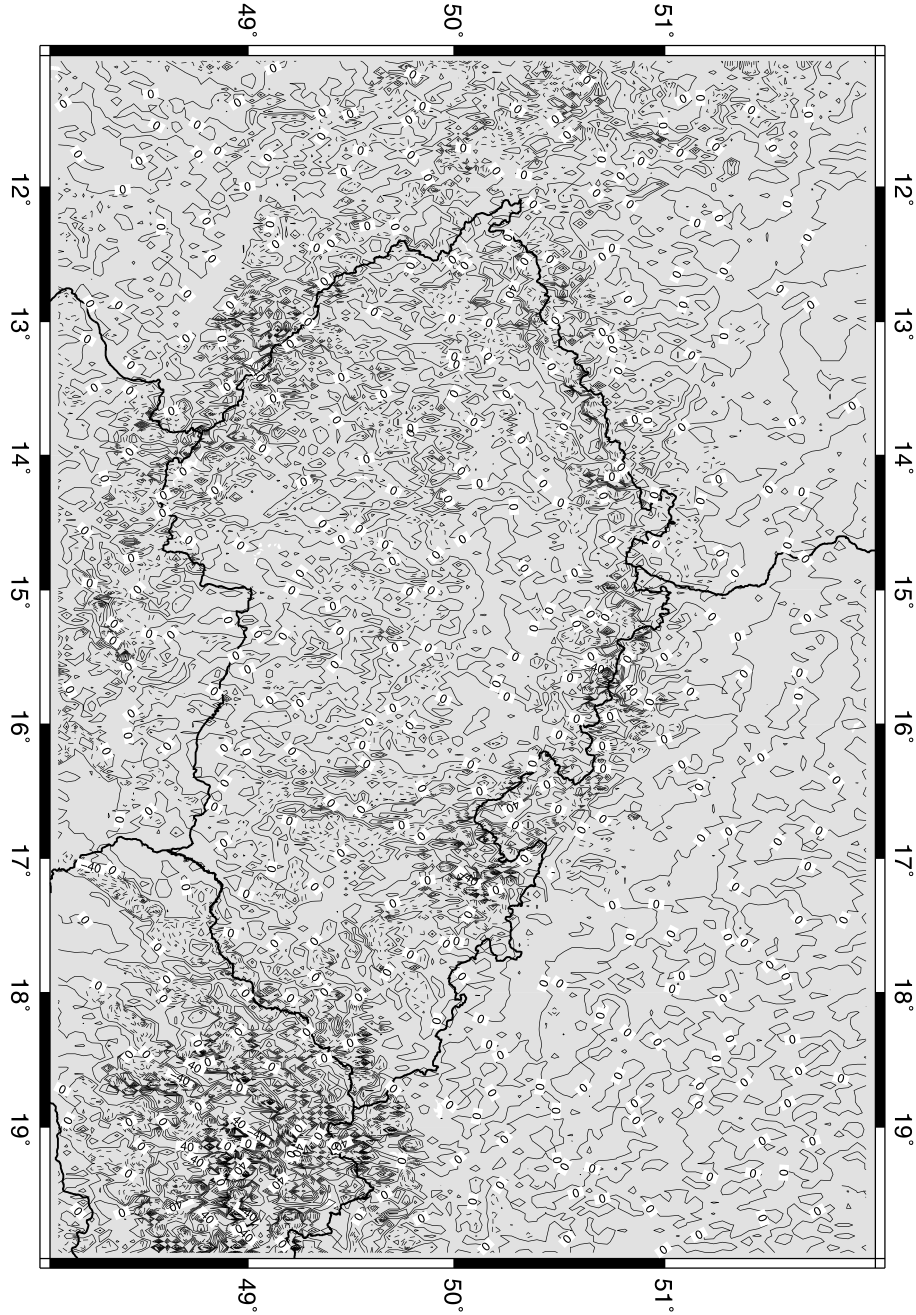

Fig. 8. Vertical gradient of the residual gravity anomaly Δg_r^ℓ (E).

vertical gradients of anomalous gravity were computed over the selected area in Central Europe. High resolution surface gravity and elevation data were used for evaluation of their residual (high-frequency) components. These values are the main output of this contribution due to their eventual link to local features of the Earth's gravity field in the area. The GGM was then used for evaluation of the reference (low-frequency) components through the spherical harmonic synthesis of the Stokes coefficients of the geopotential. Although the gravity signal cannot strictly be divided spectrally according to spatial distribution of these masses, it is mainly the high-frequency signal that may be linked to various local mass anomalies in the structure of the upper mantle. Leaving the geophysical interpretations to respective experts, this contribution was prepared with the intention of showing capabilities of classical geodetic methods in combination with the newest generation of high-resolution gravity and elevation data as well as current computer equipment capable of highly demanding numerical computations.

Despite large progress in global gravity modelling via new satellite missions dedicated to gravity field mapping (CHAMP and GRACE), local gravity and elevation data still keep their important role for estimation of local gravity field features. For areas with excellent local gravity observations (such as the area used in this contribution), interesting information on the local structure of the gravitational field caused namely by shallow mass anomalies can be extracted by gravity inversion. Despite large progress in theory and numerical techniques used for gravity inversion, input data (gravity, elevation, mass densities) are still essential for obtaining correct values of computed parameters. While there have been only relatively small advances in collection of new ground gravity data in the computation area, new elevation and mass density models are being compiled to be used in future computations.

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