Generalized geoidal estimators for deterministic modifications of spherical Stokes' function

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Abstract: Stokes' integral, representing a surface integral from the product of terrestrial gravity data and spherical Stokes' function, is the theoretical basis for the modelling of the local geoid. For the practical determination of the local geoid, due to restricted knowledge and availability of terrestrial gravity data, this has to be combined with the global gravity model. In addition, the maximum degree and order of spherical harmonic coefficients in the global gravity model is finite. Therefore, modifications of spherical Stokes' function are used to obtain faster convergence of the spherical harmonic expansion. Decomposition of Stokes' integral and modifications of Stokes' function have been studied by many geodesists. In this paper, the proposed deterministic modifications of spherical Stokes' function are generalized. Moreover, generalized geoidal estimators, when the Stokes' integral is decomposed in to spectral and frequency domains, are introduced. Higher derivatives of spherical Stokes' function and their numerical stability are discussed. Filtering and convergence properties for deterministic modifications of the spherical Stokes' function in the form of a remainder of the Taylor polynomial are studied as well.

Key words: Stokes' integral, truncated integration, deterministic modifications, geoidal estimators

1. Introduction

In the context of effective measurements using Global Navigation Satellite Systems (GNSS), the knowledge of the geoidal surface still remains a challenging problem for geoscientists all over the world. The Stokes' integral (a surface convolution integral from the product of terrestrial gravity data and the spherical Stokes' function) represents the mathematical basis for the determination of the local geoid. However, the direct application of Stokes' integral is restricted due to the lack of terrestrial gravity data. In practical determination of the local geoid, the Stokes' integral is decomposed into two parts, the truncated integration and the truncated series of spherical harmonics.

The spherical Stokes' function in the Stokes' integral plays a role of an integration kernel. Its value depends on the distance between the computation point and integration element on the surface of a reference sphere. Naturally, behaviour of the spherical Stokes' function is significant for the computation of the truncated integration. In addition, spherical Stokes' function affects the truncated series of spherical harmonics by means of spectral weights. In order to reduce the amplitudes of spectral weights, modifications of spherical Stokes' function (hereinafter referred to as modifications) have been studied by many authors.

Modifications focusing on faster convergence of the truncated series of spherical harmonics are called deterministic modifications. Mathematical principles of the deterministic modifications have been studied by *Molodensky et al. (1962); Wong and Gore (1969); Meissl (1971); Jekeli (1980, 1981); Heck and Grüninger (1987); Vaníček and Kleusberg (1987); Vaníček and Sjöberg (1991); Featherstone et al. (1998); Evans and Featherstone (2000).* Furthermore, application of least squares principles to modify spherical Stokes' function has been proposed by *Wenzel (1982), Sjöberg (1984, 1991), Sjöberg and Hunegnaw (2000)*, introducing the group of stochastic modifications. In this case also the stochastic properties of terrestrial gravity data and spherical harmonic coefficients of the global gravity model (GGM) have to be taken into account. However, the detailed formulation of modifications depends on the approach used for the decomposition of the Stokes' integral. For example, deterministic modifications together with the remove-compute-restore technique are discussed in *Featherstone et al. (1998), Vaníček and Featherstone (1998), Evans and Featherstone (2000), Featherstone (2003).* On the other hand, decomposition of the Stokes' integral in space domain only, when the reference gravity field is generated by a reference ellipsoid, is considered in *Jekeli (1980, 1981), Sjöberg and Hunegnaw (2000), Sjöberg (2003).*

Formal similarity of modifications is a motivation for their generalized expression. Consequently, the concept of generalization of geoidal estimators can be easily applied. *Sjöberg (2003)* formulated a general model for geoid estimators in the case of deterministic and stochastic modifications.

For this purpose relatively simple integration kernel has been chosen. In the present paper more complicated integration kernel proposed in *Šprlák (2008b)* is used. For the sake of simplicity, only the concept of the generalization of deterministic modifications and corresponding geoidal estimators is proposed. In Section 2 two most common approaches for decomposition of Stokes' integral are described considering spherical Stokes' function. Also the error and global mean square error of corresponding geoidal estimators are introduced. Universal expression for deterministic modifications of spherical Stokes' function is presented in Section 3. In addition, the most cited deterministic modifications of spherical Stokes' function are resolved. Numerical stability for higher derivatives of spherical Stokes' function is discussed. In Section 4 general geoidal estimators are derived. An example of filtering and convergence properties for the modifications in the form of Taylor polynomial remainder is presented. Significant results of the present paper are emphasized in conclusions.

2. Geoidal estimators with spherical Stokes' function

Stokes' integral is a well known formula for determination of the geoidal height N. From a mathematical point of view it corresponds to a surface convolution integral over a unit sphere σ in the form *(Hofmann-Wellenhof and Moritz, 2005, Eq. 2-307)*:

$$N = \frac{c}{2\pi} \iint\limits_{\sigma} \Delta g \; S(y) \; \mathrm{d}\sigma, \tag{1}$$

where $c = \dfrac{R}{2\gamma}$, R is the radius of a reference sphere, γ is the normal gravity at the surface of a reference ellipsoid,[1] $S(y)$ is the spherical Stokes' function and Δg is the gravity anomaly for which an expression in a series of spherical

[1] Originally, Eq. (1) is based on Bruns' formula $N = T/\gamma$, see e.g. *(Hofmann-Wellenhof and Moritz, 2005, Eq. 2-237)*, where T is the disturbing potential at the surface of the geoid and γ is defined above. However the surface of the geoid is approximated by a reference sphere with radius R. Therefore the integration is performed over a reference sphere and the normal gravity γ is defined at the surface of a reference ellipsoid.

harmonics is *(Hofmann-Wellenhof and Moritz, 2005, Eq. 9-14)*:

$$\Delta g = \sum_{n=2}^{\infty} \Delta g_n = \frac{GM}{r^2} \sum_{n=2}^{\infty} (n-1) \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} \bar{P}_{nm} (\sin \varphi) \times$$

$$\times \left[\Delta \bar{C}_{nm} \cos(m\lambda) + \bar{S}_{nm} \sin(m\lambda)\right] . \qquad (2)$$

In the last equation, GM is the product of Newtonian gravitational constant and the mass of the Earth including oceans and atmosphere, (r, φ, λ) are the spherical polar coordinates of the computation point, a is the length of the semimajor axis of a reference ellipsoid, $\Delta\bar{C}_{nm}$ and $\bar{S}_{nm}$ are the fully normalised spherical harmonic coefficients of degree n and order m reduced by the corresponding coefficients of the reference gravity field of a reference ellipsoid, and $\bar{P}_{nm}(\sin\varphi)$ are the fully normalised Legendre functions of the first kind. Spectral representation of the spherical Stokes' function $S(y)$ *(Stokes, 1849)* in a series of Legendre polynomials $P_n(y)$ is:

$$S(y) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(y), \qquad (3)$$

where $y = \cos\psi$ and ψ is the spherical distance between the computation point and integration element.

As mentioned above, direct application of Stokes' integral Eq. (1) is restricted due to the lack of terrestrial gravity data. This restriction holds also for another analytical solutions of geodetic boundary value problems in the form of surface integrals, e.g. Hotine's and Poisson's integral. Usually, only very close terrestrial gravity data, several arc degrees around the computation area, are available. In addition, global integration is not reasonable because of time consuming computation and limited computer memory. For the practical determination of the geoid, geoidal estimators based on the decomposition of Stokes' integral in the space and frequency domains, are formulated. In both cases, integration of terrestrial gravity data with proper integration radius is performed. The rest of the truncated integration is expanded into a series of spherical harmonics and computed by GGM. In the next subsections, two most common approaches of decomposition of the Stokes' integral are discussed.

2.1. Decomposition of Stokes' integral in space domain

The presented procedure for solving the problem of restricted availability of gravity data and its combination with spherical harmonic coefficients of the GGM was originally proposed by *Molodensky et al. (1962)*. Review of this approach with some important aspects is also presented in *Vaníček et al. (2003)*. In order to see the main differences between the geoidal estimator with spherical Stokes' function and the general geoidal estimator derived in Section 4, let us describe the decomposition of Stokes' integral in space domain. Let us suppose that the Stokes' integral Eq. (1) is decomposed into two parts:

$$N = \frac{c}{2\pi} \iint\limits_{\sigma_0} \Delta g\, S(y)\, \mathrm{d}\sigma + \frac{c}{2\pi} \iint\limits_{\sigma-\sigma_0} \Delta g\, S(y)\, \mathrm{d}\sigma. \tag{4}$$

First term on the right hand side of Eq. (4) corresponds to a truncated integration of gravity data in domain σ_0 (termed also as the effect of the near zone). Truncated integration with integration radius ψ_0 around each computation point is computed by standard algorithms for numerical integration. Second term in Eq. (4) (called also the effect of the distant zone) represents the rest of truncated integration in the domain $\sigma - \sigma_0$. Because there are no terrestrial gravity data available in the domain $\sigma - \sigma_0$, effect of the distant zone is expanded into a series of spherical harmonics. For this purpose, let us define an error kernel $\Delta K(y)$ on the interval $-1 \leq y < 1$ with $y_0 = \cos\psi_0$ by the following equation:

$$\Delta K(y) = \begin{cases} 0, & y_0 \leq y < 1 \\ S(y), & -1 \leq y < y_0 \end{cases} \tag{5}$$

which can be expanded into a series of Legendre polynomials in the form:

$$\Delta K(y) = \sum_{n=2}^{\infty} \frac{2n+1}{2} Q_n(y_0)\, P_n(y), \tag{6}$$

where the truncation error coefficients $Q_n(y_0)$ are:

$$Q_n(y_0) = \int_{-1}^{1} \Delta K(y)\, P_n(y)\, \mathrm{d}y = \int_{-1}^{y_0} S(y)\, P_n(y)\, \mathrm{d}y. \tag{7}$$

It is important to note that neither analytical nor numerical methods for solving definite integrals are used for computation of truncation error coefficients. Recurrence relations derived in *Paul (1973)* and *Hagiwara (1976)* are preferred. Using the definition of the error kernel according to Eq. (5), the integration over a reference sphere can be performed. Supposing spectral representation of Stokes' function and its error kernel according to Eqs. (3) and (6) and using orthogonality relations for spherical harmonics, the effect of the distant zone is expressed as:

$$\frac{c}{2\pi} \iint_{\sigma-\sigma_0} \Delta g \; S(y) \, \mathrm{d}\sigma = \frac{c}{2\pi} \iint_{\sigma} \Delta g \; \Delta K(y) \, \mathrm{d}\sigma = c \sum_{n=2}^{\infty} Q_n(y_0) \; \Delta g_n . \tag{8}$$

From the last equation one can see that the magnitude of the effect of the distant zone depends on the n-th surface spherical harmonic of the gravity anomaly arising from Eq. (2) and the truncation error coefficients $Q_n(y_0)$. On the other hand, according to Eq. (7), the size of integration radius ψ_0 and behaviour of integration kernel affect the amplitudes of the truncation error coefficients.

Let us now formulate the practical geoidal estimator in which terrestrial (with superscript T) and satellite gravity data in the form of spherical harmonic coefficients of the GGM (with superscript S) are combined. Due to measurement and data reduction errors the theoretical (true, errorless) values of terrestrial gravity data Δg^T must be replaced by their estimates $\Delta \hat{g}^T$. The difference between the estimate and theoretical value represents the error $\epsilon^T = \Delta \hat{g}^T - \Delta g^T$. Similarly, the theoretical value of the gravity anomaly for n-th surface harmonic Δg_n^S is replaced by its estimate $\Delta \hat{g}_n^S$ with the error $\epsilon_n^S = \Delta \hat{g}_n^S - \Delta g_n^S$. Respecting the estimates of the gravity data in Eqs. (4) and (8) the geoidal estimator has the form:

$$\hat{N} = \frac{c}{2\pi} \iint_{\sigma_0} \Delta \hat{g}^T \; S(y) \, \mathrm{d}\sigma + c \sum_{n=2}^{M_{max}} Q_n(y_0) \; \Delta \hat{g}_n^S , \tag{9}$$

where M_{max} is the maximum degree of spherical harmonic coefficients of the GGM. The geoidal estimator Eq. (9) consists of two analogous terms. The first term corresponds to a truncated integration in a space domain in which the significance of terrestrial gravity data is determined by the spherical Stokes' function $S(y)$. In the second term, weight of satellite gravity data in a truncated series is determined by truncation error coefficients $Q_n(y_0)$ or, in general, by spectral weights which are related to spherical Stokes' function through Eq. (7). Evidently, a reference gravity field is generated by a reference ellipsoid. Therefore Eq. (9) can be called the geoidal estimator with the reference gravity field generated by a reference ellipsoid. Using the difference between the geoidal estimator Eq. (9) and the Stokes' integral Eq. (1) the error of geoidal estimator is defined as follows:

$$\epsilon_{\hat{N}} = \hat{N} - N = c \sum_{n=2}^{\infty} \left[\frac{2}{n-1} - Q_n(y_0) \right] \epsilon_n^T + c \sum_{n=2}^{M_{max}} Q_n(y_0) \epsilon_n^S - - c \sum_{n=M_{max}+1}^{\infty} Q_n(y_0) \Delta g_n^T . \tag{10}$$

The symbols Δg_n^T and ϵ_n^T are the spectral components of the theoretical values for terrestrial gravity data and their corresponding errors. Propagation of terrestrial and satellite gravity data errors are defined by the first and second terms in Eq. (10). Omission of higher degree spherical harmonics above the maximum degree M_{max} of the spherical harmonic coefficients of the GGM is defined by the third term in Eq. (10). Evidently, all terms are controlled by the spectral weights depending on the integration kernel used in truncated integration.

2.2. Remove-compute-restore technique

In the previous subsection the gravity anomalies with all their frequencies are integrated over domain σ_0. Let us suppose that the low frequencies are removed up to degree M in the first term of Eq. (4), i.e. the reference gravity field is generated by a spheroid to degree M. In this way also the decomposition in spectral domain is performed and the effect of the near zone is:

$$\frac{c}{2\pi} \iint_{\sigma_0} \Delta g \; S(y) \, \mathrm{d}\sigma = \frac{c}{2\pi} \iint_{\sigma_0} \left[\Delta g - \sum_{n=2}^{M} \Delta g_n \right] \; S(y) \, \mathrm{d}\sigma + + c \sum_{n=2}^{M} \left[\frac{2}{n-1} - Q_n(y_0) \right] \; \Delta g_n . \tag{11}$$

Substituting Eq. (11) into Eq. (4), considering the expression for the effect of the distant zone Eq. (8) and $M = M_{max}$, a geoidal estimator, widely used in practical determination of the geoid, known as the remove-compute-restore (RCR) technique *Rapp and Rummel (1975); Sansó (2005)* can be derived in the form:

$$\hat{N}^{RCR} = c \sum_{n=2}^{M_{max}} \frac{2}{n-1} \Delta\hat{g}_n^S + \frac{c}{2\pi} \iint\limits_{\sigma_0} \left[\Delta\hat{g}^T - \sum_{n=2}^{M_{max}} \Delta\hat{g}_n^S \right] S(y)\, \mathrm{d}\sigma. \tag{12}$$

First term on the right-hand side represents a low frequency geoid which is computed by the GGM. High frequencies of the geoid are obtained by truncated integration of residual gravity anomalies in the second term of Eq. (12). From a mathematical point of view, RCR technique is equivalent to the geoidal estimator Eq. (9). Therefore, the corresponding error of RCR technique is defined by Eq. (10). In comparison to geoidal estimator Eq. (9), *Sjöberg and Hunegnaw (2000)* considered RCR technique as time consuming and disadvantageous from a data administration point of view. For more detailed discussion about RCR technique and geoidal estimator Eq. (9) see also *Sjöberg (2005), Ellmann (2005).*

2.3. The global mean square error

The error of geoidal estimator according to Eq. (10) depends on the position of the computation point. For an investigation of significant properties (e.g. convergence and filtering properties, see also Section 4) of geoidal estimators more appropriate quantity is the global mean square error (GMSE). It is defined as a square root of average, over the sphere, of the squared error of geoidal estimator *(Jekeli, 1981).* For a square value of GMSE of the geoidal estimators Eqs. (9) and (12), it takes the form:

$$m_{\hat{N}}^2 = \frac{1}{4\pi} \iint\limits_{\sigma} \epsilon_{\hat{N}}^2 \, \mathrm{d}\sigma = c^2 \sum_{n=2}^{\infty} \left[\frac{2}{n-1} - Q_n(y_0) \right]^2 \sigma_n^2 +$$

$$+ c^2 \sum_{n=2}^{M_{max}} \left[Q_n(y_0) \right]^2 \mathrm{d}c_n + c^2 \sum_{n=M_{max}+1}^{\infty} \left[Q_n(y_0) \right]^2 c_n. \tag{13}$$

Terrestrial and satellite gravity anomaly error degree variances σ_n^2 and $\mathrm{d}c_n$, respectively, and gravity anomaly degree variances c_n are defined as follows *(Sjöberg, 2003)*:

$$\sigma_n^2 = \frac{1}{4\pi} \iint\limits_{\sigma} \left(\epsilon_n^T\right)^2 \, \mathrm{d}\sigma, \tag{14}$$

$$\mathrm{d}c_n = \frac{1}{4\pi} \iint\limits_{\sigma} \left(\epsilon_n^S\right)^2 \, \mathrm{d}\sigma, \tag{15}$$

$$c_n = \frac{1}{4\pi} \iint\limits_{\sigma} \left(\Delta g_n^T\right)^2 \, \mathrm{d}\sigma. \tag{16}$$

In practical applications, series of spherical harmonic coefficients and their standard errors, which are available in the current GGMs, are used to compute the quantities c_n and $\mathrm{d}c_n$, while σ_n^2 is estimated from a covariance function, see e.g. *Ellmann (2005)*.

3. Deterministic modifications of spherical Stokes' function

In the previous Section, relation between the truncation error coefficients, integration kernel and truncation error was demonstrated by Eqs. (7) and (8). In practice, truncation error is computed by a series of spherical harmonics and spherical harmonic coefficients of the GGM. Since the maximum degree and order of the spherical harmonic coefficients is limited, several approaches in the form of deterministic modifications of the spherical Stokes' function to obtain more rapid convergence of the truncation error have been suggested. It should be pointed out that deterministic modifications can be generalized by the following equation:

$$\tilde{S}_B(y) = \tilde{S}(y) - \sum_{b=0}^{B} \frac{(y-y_0)^b}{b!} \frac{\mathrm{d}^b \tilde{S}(y_0)}{\mathrm{d}y^b}, \quad y_0 \leq y < 1. \tag{17}$$

The first term in Eq. (17) is:

$$\tilde{S}(y) = S(y) - \sum_{k=2}^{P} \frac{2k+1}{2} \, a_k \, P_k(y) - \sum_{k=2}^{L} \frac{2k+1}{2} \, b_k \, P_k(y), \tag{18}$$

where a_k and b_k represent the first and second modification coefficients. From the mathematical point of view, the general integration kernel $\tilde{S}_B(y)$ corresponds to the remainder of a Taylor polynomial to degree B at the cosine of integration radius *(Stein, 1987)*. Therefore, when only deterministic modifications with $B \geq 0$ are discussed, the term modifications in the form of Taylor polynomial remainder will be used throughout the text.

Most cited deterministic modifications defined in Table 1 are distinguished by a proper choice of modification coefficients a_k, b_k and degree B in Eqs. (17) and (18). In the first place, spherical Stokes' function is specified when $a_k = b_k = 0$. The concept of deterministic modifications was formerly proposed by *Molodensky et al. (1962)*. Their deterministic modification is based on a minimization of the L_2-norm of the truncation error. Therefore, one can say that deterministic modification by *Molodensky et al. (1962)* has a mathematical criterion from which modification coefficients b_k to degree L arise from the system of linear equations. Conditioning of the system of linear equations depends on the integration radius, degree L and the presence of modification coefficients b_0 and b_1, for more details see *Sjöberg and Hunegnaw (2000), Featherstone (2003)*. Deterministic modification by *Molodensky et al. (1962)* together with its possible alternatives are discussed in *Jekeli (1980, 1981)*.

Wong and Gore (1969) proposed an approach with modification coefficients $a_k = 2/(k-1)$. According to Eq. (3), long wavelength part of spherical Stokes' function is removed up to degree P. In this way a spheroidal Stokes' function is defined. The same kernel corresponds to the analytical solution of the Stokes' boundary value problem for a higher degree reference gravity field, see *Vaníček and Sjöberg (1991)*. Combination of the previous two deterministic modifications was proposed by *Vaníček and Kleusberg (1987)*. The corresponding modification coefficients b_k are evaluated from a system of linear equations whose numerical stability depends on integration radius and on degrees L and P. It is important to note that modification coefficients b_k for deterministic modification by *Molodensky et al. (1962)* and *Vaníček and Kleusberg (1987)* differ from each other. However, in a special case when $L = P$, modification coefficients are equal and both kernels are equivalent, see *Vaníček and Sjöberg (1991), Sjöberg and Featherstone (2004)*. Note that the described deterministic modifications, as well as spherical Stokes' function, have discontinuity of the corresponding error

Table 1. Spherical Stokes' function and its deterministic modifications[2]

Modification	B	a_k	b_k
Stokes (1849)	–	0	0
Molodensky et al. (1962)	–	0	$\sum_{k=2}^{L} \frac{2k+1}{2}\, b_k\, e_{nk} = Q_n(y_0)$
Wong and Gore (1969)	–	$\frac{2}{k-1}$	0
Vaníček and Kleusberg (1987)	–	$\frac{2}{k-1}$	$\sum_{k=2}^{L} \frac{2k+1}{2}\, b_k\, e_{nk} = Q_n(y_0) - \sum_{k=2}^{P} \frac{2k+1}{2}\, a_k\, e_{nk}$
Meissl (1971)	0	0	0
Jekeli (1980)	0	0	$\sum_{k=2}^{L} \frac{2k+1}{2}\, b_k\, e_{nk} = Q_n(y_0)$
Heck and Grüninger (1987)	0	$\frac{2}{k-1}$	0
Featherstone et al. (1998)	0	$\frac{2}{k-1}$	$\sum_{k=2}^{L} \frac{2k+1}{2}\, b_k\, e_{nk} = Q_n(y_0) - \sum_{k=2}^{P} \frac{2k+1}{2}\, a_k\, e_{nk}$

kernel at y_0, see Eq. (5).

On the contrary, the continuity of an error kernel at y_0 is an attribute for modifications in the form of Taylor polynomial remainder. Using the properties of orthogonal series expansions more rapid convergence for the amplitudes of the truncation error coefficients and consequently the truncation error is achieved without the direct criterion of minimization. Reduced magnitude of the truncation error near zeros of spherical Stokes' function was observed by *de Witte (1967)*. *Meissl (1971)* proposed an integration kernel in the form of algebraic subtraction of spherical Stokes' function and its value at integration radius. *Heck and Grüninger (1987)* used the same idea for deterministic modification by *Wong and Gore (1969)*. Moreover, *Jekeli (1980)* applied this principle to deterministic modification by *Molodensky et al. (1962)* and *Featherstone et al. (1998)* for deterministic modification by *Vaníček and Kleusberg (1987)*. *Evans and Featherstone (2000)* considered continuous error kernels with their derivatives up to an arbitrary order for

[2] Integrals $e_{nk} = \int_{-1}^{y_0} P_n(y)\, P_k(y)\, dy$ are termed Paul's coefficients. Recurrence formulas for their numerical computation are given in *Paul (1973), Jekeli (1980)*.

spherical Stokes' function and deterministic modifications by *Molodensky et al. (1962), Wong and Gore (1969), Vaníček and Kleusberg (1897).*

For modifications in the form of Taylor polynomial remainder, a problem of numerical stability for higher derivatives of the kernel $\tilde{S}(y)$ arises. For the sake of simplicity, let us demonstrate the behaviour of higher derivatives of spherical Stokes' function only up to the 4th order which have been derived in *Šprlák (2008a).* From Fig. 1 one can see an increasing magnitude of the absolute values with increasing order of the derivative. When the spherical distance is decreasing, i.e. $y \to 1$, a higher growth can been seen. For the derivatives of the 3rd and 4th order, the magnitudes of orders more than 10^{15} are reached as $y \to 1$. Because of higher frequency of amplitudes for deterministic modifications by *Molodensky et al. (1962), Wong and Gore (1969), Vaníček and Kleusberg (1897)*, larger values for their derivatives can be expected. Numerical experiments in *Šprlák (2008b)* showed that stable computation of GMSE is guaranteed for degrees $B \leq 2$.

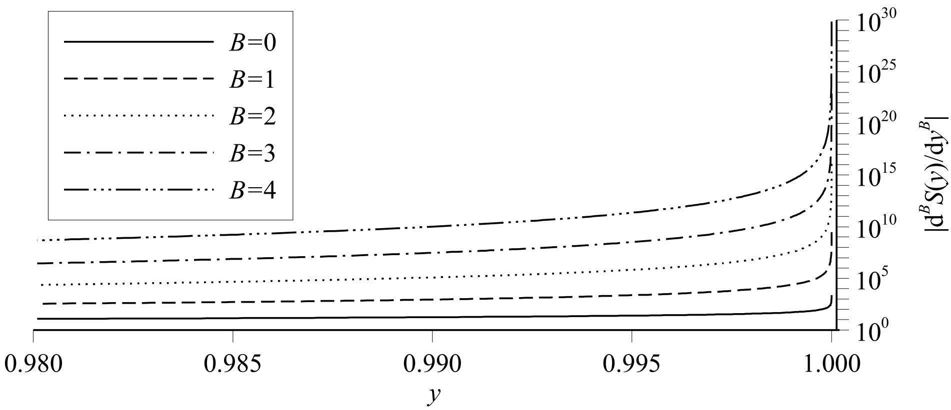

Fig. 1. Absolute value of spherical Stokes' function and its derivatives up to the 4th order (logarithmic scale on vertical axis).

4. General geoidal estimators for deterministic modifications

General geoidal estimators can be formulated considering a universal expression of deterministic modifications. Substituting the integration kernel

Eq. (17) into Stokes' integral Eq. (1) and after some manipulation, the following equation for geoidal height is obtained:

$$N = \frac{c}{2\pi} \iint_{\sigma_0} \Delta g \ \tilde{S}_B(y) \ \mathrm{d}\sigma + \frac{c}{2\pi} \iint_{\sigma-\sigma_0} \Delta g \ \tilde{S}(y) \ \mathrm{d}\sigma +$$

$$+ \frac{c}{2\pi} \iint_{\sigma_0} \Delta g \sum_{b=0}^{B} \frac{(y-y_0)^b}{b!} \frac{\mathrm{d}^b \tilde{S}(y_0)}{\mathrm{d}y^b} \ \mathrm{d}\sigma +$$

$$+ \frac{c}{2\pi} \left[\sum_{k=2}^{P} \frac{2k+1}{2} \ a_k + \sum_{k=2}^{L} \frac{2k+1}{2} \ b_k \right] \iint_{\sigma} P_k(y) \sum_{n=2}^{\infty} \Delta g_n \ \mathrm{d}\sigma. \tag{19}$$

Note that Eq. (19) is equivalent to the Stokes' integral (1) which can be easily proved. Also note that *Sjöberg (2003)* considered only the first two terms in Eq. (19) to formulate the general geoidal estimator. The first term in Eq. (19) represents the effect of the near zone and as we know from the previous Section it is computed by standard algorithms for numerical integration. Let us now define an error kernel $\Delta\tilde{K}_B(y)$ for general integration kernel in the interval $-1 \le y < 1$ by the equation:

$$\Delta\tilde{K}_B(y) = \begin{cases} \sum\limits_{b=0}^{B} \dfrac{(y-y_0)^b}{b!} \dfrac{\mathrm{d}^b \tilde{S}(y_0)}{\mathrm{d}y^b}, & y_0 \le y < 1 \\ \tilde{S}(y), & -1 \le y < y_0 \end{cases} \tag{20}$$

from which $\Delta\tilde{K}_B(y_0) = \tilde{S}(y_0)$ follows. Moreover, we consider that the error kernel is continuous up to its derivatives of B-th order at y_0. In other words, the error kernel $\Delta\tilde{K}_B(y)$ corresponds to a smooth continuation of $\tilde{S}(y)$ from the interval $-1 \le y < y_0$ into the interval $y_0 \le y < 1$ by Taylor polynomial expansion at y_0. Using the definition of the error kernel $\Delta\tilde{K}_B(y)$ by Eq. (20), for the second and third terms in Eq. (19) we have

$$\frac{c}{2\pi} \iint_{\sigma-\sigma_0} \Delta g \ \tilde{S}(y) \ \mathrm{d}\sigma + \frac{c}{2\pi} \iint_{\sigma_0} \Delta g \sum_{b=0}^{B} \frac{(y-y_0)^b}{b!} \frac{\mathrm{d}^b \tilde{S}(y_0)}{\mathrm{d}} y^b \ \mathrm{d}\sigma =$$

$$= \frac{c}{2\pi} \iint_{\sigma} \Delta g \ \Delta\tilde{K}_B(y) \ \mathrm{d}\sigma = c \sum_{n=2}^{\infty} \tilde{Q}_n^B(y_0) \ \Delta g_n. \tag{21}$$

The error kernel $\Delta\tilde{K}_B(y)$ can be expanded, in a same way as in Eq. (6), into a series of Legendre polynomials where the corresponding error coefficients $\tilde{Q}_n^B(y_0)$ are:

$$\tilde{Q}_n^B(y_0) = \int_{-1}^{1} \Delta\tilde{K}_B(y)\; P_n(y)\;\mathrm{d}y =$$

$$= \int_{-1}^{y_0} \tilde{S}(y)\; P_n(y)\;\mathrm{d}y + \int_{y_0}^{1} \sum_{b=0}^{B} \frac{(y-y_0)^b}{b!} \frac{\mathrm{d}^b \tilde{S}(y_0)}{\mathrm{d}y^b}\; P_n(y)\;\mathrm{d}y. \tag{22}$$

Alternative relations suitable for practical computation of truncation error coefficients $\tilde{Q}_n^B(y_0)$ for $B \leq 2$ are given in *Šprlák (2008a,b).* The fourth term in Eq. (19) is expressed using a well known integral identity for the n-th surface spherical harmonics of gravity anomaly in the form *(Hofmann-Wellenhof and Moritz, 2005, Eq. 1-89)*:

$$\Delta g_n = \frac{2n+1}{4\pi} \iint_{\sigma} P_n(y)\; \sum_{n=2}^{\infty} \Delta g_n \;\mathrm{d}\sigma. \tag{23}$$

Then the fourth term is:

$$\frac{c}{2\pi} \left[\sum_{k=2}^{P} \frac{2k+1}{2}\; a_k + \sum_{k=2}^{L} \frac{2k+1}{2}\; b_k \right] \iint_{\sigma} P_k(y)\; \sum_{n=2}^{\infty} \Delta g_n \;\mathrm{d}\sigma =$$

$$= \; c \sum_{n=2}^{P} a_n\; \Delta g_n + c \sum_{n=2}^{L} b_n\; \Delta g_n. \tag{24}$$

Consequently the general geoidal estimator with the reference gravity field generated by the reference ellipsoid can be formulated. Assuming Eqs. (21) and (24) in (19), considering estimates of the gravity anomalies and a maximum degree M_{max} of spherical harmonic coefficients of the GGM, the resulting equation is:

$$\hat{N}_B = \frac{c}{2\pi} \iint_{\sigma_0} \Delta\hat{g}^T\; \tilde{S}_B(y)\;\mathrm{d}\sigma + c \sum_{n=2}^{M_{max}} \left[d_n + \tilde{Q}_n^B(y_0) \right]\; \Delta\hat{g}_n^S, \tag{25}$$

where

$$d_n = \begin{cases} a_n + b_n, & \text{if} \quad 2 \le n \le L \\ a_n, & \text{if} \quad L < n \le P \\ 0, & \text{if} \quad P < n < \infty \end{cases} \tag{26}$$

Comparing Eqs. (9) and (25), a formal similarity of both geoidal estimators can be seen. Evidently the transition of integration kernel $S(y)$ into $\tilde{S}_B(y)$ in truncated integration causes the change of truncation error coefficients $Q_n(y_0)$ into spectral weights $[d_n + \tilde{Q}_n^B(y_0)]$ in truncated series of spherical harmonics. In the case of RCR technique only the integration kernel in truncated integration is changed, then:

$$\hat{N}_B^{RCR} = c \sum_{n=2}^{M_{max}} \frac{2}{n-1} \Delta\hat{g}_n^S + \frac{c}{2\pi} \iint_{\sigma_0} \left[\Delta\hat{g}^T - \sum_{n=2}^{M_{max}} \Delta\hat{g}_n^S \right] \tilde{S}_B(y) \, \mathrm{d}\sigma. \tag{27}$$

Eq. (27) is equivalent to the geoidal estimator Eq. (25). The corresponding error of general geoidal estimators Eqs. (25) and (27) is defined as follows:

$$\epsilon_{\hat{N}_B} = \hat{N}_B - N = c \sum_{n=2}^{\infty} \left[\frac{2}{n-1} - d_n - \tilde{Q}_n^B(y_0) \right] \epsilon_n^T +$$

$$+ c \sum_{n=2}^{M_{max}} \left[d_n + \tilde{Q}_n^B(y_0) \right] \epsilon_n^S - c \sum_{n=M_{max}+1}^{\infty} \tilde{Q}_n^B(y_0) \, \Delta g_n^T \tag{28}$$

and the GMSE is:

$$m^2_{\hat{N}_B} = \frac{1}{4\pi} \iint_{\sigma} \epsilon^2_{\hat{N}_B} \, \mathrm{d}\sigma = c^2 \sum_{n=2}^{\infty} \left[\frac{2}{n-1} - d_n - \tilde{Q}_n^B(y_0) \right]^2 \sigma_n^2 +$$

$$+ c^2 \sum_{n=2}^{M_{max}} \left[d_n + \tilde{Q}_n^B(y_0) \right]^2 \mathrm{d}c_n + c^2 \sum_{n=M_{max}+1}^{\infty} \left[\tilde{Q}_n^B(y_0) \right]^2 c_n. \tag{29}$$

Modifying the spherical Stokes' function, the error of general geoidal estimators and their GMSE are also affected. Comparing Eqs. (10) and (13) with (28) and (29) truncation error coefficients $Q_n(y_0)$ are replaced by their spectral counterparts $[d_n + \tilde{Q}_n^B(y_0)]$ in the first two terms representing the propagation of errors from terrestrial gravity data and spherical harmonic coefficients of the GGM. The omission of higher degree spherical harmonics is controlled by the truncation error coefficients $\tilde{Q}_n^B(y_0)$. According to the amplitudes of spectral weights, the significant filtering and convergence properties of integration kernels have been studied, see *Vaníček and Featherstone (1998)*. Note also that the stochastic modifications of the spherical Stokes' function are based on a minimization of the GMSE.

As an illustrative example, let us inspect the behaviour of all terms defining GMSE for modifications in the form of Taylor polynomial remainder. For the sake of simplicity the case of $a_n = b_n = d_n = 0$ will be considered. In Fig. 2a the decreasing magnitude of amplitudes of truncation error coefficients $\tilde{Q}_n^B(y_0)$ with increasing B is proved for $n < 150$. For a higher n negligible differences between the amplitudes can be seen. Therefore when the maximum degree M_{max} of spherical harmonic coefficients in the GGM is sufficiently high only negligible differences of the third term in Eqs. (28) and (29) can be observed. Moreover, *Evans and Featherstone (2000)* proved

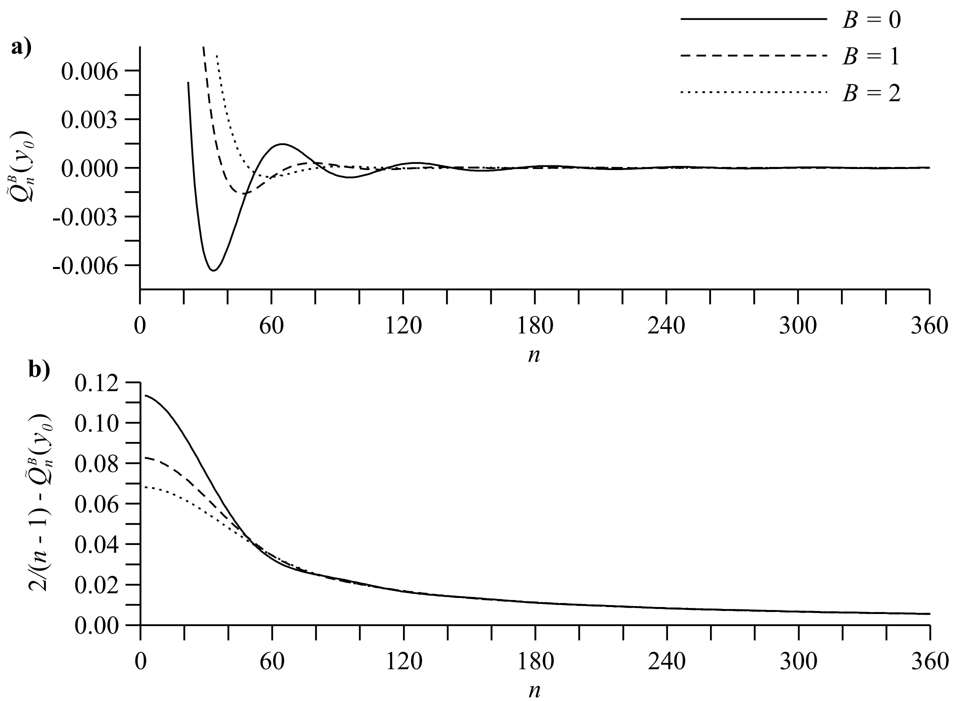

Fig. 2. Graphical representation of: **a)** $\tilde{Q}_n^B(y_0)$, $n \geq 2$, **b)** $2/(n-1) - \tilde{Q}_n^B(y_0)$, $n \geq 2$; $\psi_0 = 6^\circ$, $a_k = b_k = d_n = 0$.

that the convergence of the truncation error coefficients $\tilde{Q}_n^B(y_0)$ for an odd value of B is the same as for the previous even B. This is in contrast to graphical representation in Fig. 2a. On the other hand, according to *Featherstone (2003)* the value of truncation error coefficients $\tilde{Q}_n^B(y_0)$ with increasing B is not reduced for all n. Until the oscillation is observed, the slower decrease of truncation error coefficients $\tilde{Q}_n^B(y_0)$ with increasing B is visible. The second term in Eqs. (28) and (29) is affected and the errors of the low degree spherical harmonic coefficients of the GGM are more significant when B is higher. The first term in Eqs. (28) and (29) represents the effect of the propagation from the errors of terrestrial gravity data. In Fig. 2b the reduced values of coefficients $2/(n-1) - \tilde{Q}_n^B(y_0)$ for $n \leq 50$ with increasing B are shown. Therefore the low degree errors of terrestrial gravity data are reduced when B is higher. Modifications in the form of Taylor polynomial remainder are significant not only because of rapid convergence of the truncation error but also for their filtering properties.

5. Conclusions

In the present paper the general geoidal estimators using deterministic modifications of the spherical Stokes' function have been discussed. At first two methods for the decomposition of Stokes' integral are shown, combining terrestrial gravity data and spherical harmonic coefficients of the GGM. Decomposition in space domain only leads to the geoidal estimator with reference gravity field generated by a reference ellipsoid. The mathematically equivalent geoidal estimator represents the RCR technique when the decomposition is applied also in frequency domain. The corresponding error and the GMSE of geoidal estimators is affected by the propagation of errors from terrestrial gravity data, spherical harmonic coefficients of the GGM, ommision error and behavior of the integration kernel.

Although the mathematical principles of deterministic modifications differ, their formal similarity motivates the universal expression. Using Eqs. (17) and (18), the properly chosen modification coefficients a_k, b_k and the degree of Taylor polynomial B, most cited deterministic modifications can be resolved. Moreover, when $B \geq 0$, the modifications in the form of Taylor polynomial remainder are defined. Due to the numerical instability of higher derivatives of the Stokes' function, small values of B should be preferred. Considering the universal expression for deterministic modifications, the general geoidal estimators have been formulated. A change of integration kernel in the truncated integration for geoidal estimator with reference gravity field generated by a reference ellipsoid causes a change of spectral weights in truncated series of spherical harmonics. In the case of RCR technique, only the modified integration kernel is considered in the truncated integration. Significant properties for modifications in the form of Taylor polynomial remainder when $a_n = b_n = d_n = 0$ have been investigated. Reduced magnitudes of the truncation error coefficients $\tilde{Q}_n^B(y_0)$ have been demonstrated for subsequently increasing values of B. It has been shown that with increasing value of B the GMSE of geoidal estimators is more affected by propagation of errors from the spherical harmonic coefficients of the GGM. On the other hand, a low degree errors of terrestrial gravity data are reduced considering the higher values of B.

The advantage of generalization proposed for deterministic modifications of Stokes' function can be applied in an analogous manner to different integration kernels, e.g. Hotine's or Poisson's functions. Moreover, it will be useful to extend the proposed generalization for stochastic modifications. For this purpose the minimization of GMSE for modifications in the form of Taylor polynomial remainder should be examined, forming a base for the future work.

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