Nonlinear magneto-convection due to compositional and thermal buoyancy with Soret effect

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A b s t r a c t: The linear stability of magnetoconvection due to compositional and thermal buoyancy has been investigated. We have obtained the values of Takens-Bogdanov bifurcation points by plotting graphs of neutral curves corresponding to stationary and oscillatory convection for different values of physical parameters relevant to magnetoconvection in Earth's outer core. We have derived a nonlinear two-dimensional Landau-Ginzburg equation with real coefficients near the onset of stationary convection at a supercritical pitchfork bifurcation and two nonlinear two-dimensional coupled Landau-Ginzburg type equations with complex coefficients near the onset of oscillatory convection at a supercritical Hopf bifurcation. We have discussed the stability regions of standing and travelling waves. We have also discussed the occurence of secondary instabilities such as Eckhaus, zigzag and Benjamin-Feir instabilities.

Key words: weakly non-linear analysis, thermohaline magnetoconvection, Earth's outer core

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1. Introduction

Recent developments, both theoretical and experimental, have stimulated widespread interest in the problem of thermal and compositional magneto-convection in the Earth's outer core. When a molten two-component fluid with the heavier component having the higher melting point is chilled from below, the heavy component solidifies (*Hills et al., 1983*). The main source of heat in the Earth's outer core is cooling, boosted by latent heat of freezing of the liquid outer core onto the solid inner core. Thus the Earth's outer core is stirred by both thermal and compositional convection (*Loper, 2000*). Due to the Earth's magnetic field, we must investigate the problem of thermal and compositional magneto-convection in the Earth's outer core. This problem of thermal and compositional magneto-convection in two-component fluid (liquid iron and e.g. liquid sulphur) is similar to the problem of thermohaline magneto-convection except for the fact that a temperature difference can drive a mass current. *Tagare et al. (2006)* studied the problem of rotating compositional and thermal convection in the Earth's outer core.

In this paper we have considered the problem of thermal and compositional magneto-convection by considering the contribution of material diffusivity. Owing to the two component nature of the fluid one has Soret effect and this leads to an additional control parameter ψ (separation ratio) besides thermal Rayleigh number R. Thermohaline convection, magneto-convection and thermal and compositional convection in the Earth's outer core are examples of double diffusive systems. In thermohaline convection, the temperature and saline concentration provide two diffusivities. In magneto-convection, the temperature and magnetic field provide the two diffusivities. In thermal and compositional convection, the temperature and concentration of lighter component of fluid in a two-component fluid provide two diffusivities. *Tagare et al (2001)* studied the problem of thermohaline magnetoconvection in the Earth's outer core. Thermohaline and compositional magneto-convection is an example of triple diffusive system, where temperature, magnetic field and concentration of lighter component of fluid in a two-component fluid provide three diffusivities.

In section 2, we write basic equations. In section 3, we perform the linear stability analysis. Since the bifurcation is a continuous one, only a slow modulation of the convective roll pattern is allowed by the fluid equations near the onset. The time evolution of general pattern is developed by means of a multiple scale analysis used by *Newell and Whitehead (1969)* and *Segel (1969)* near the onset of stationary convection at a supercritical pitchfork bifurcation and convection at a supercritical Hopf bifurcation. In section 4, we have derived a nonlinear two-dimensional Landau-Ginzburg equation in complex amplitude $A(X,Y,T)$ with real coefficients near a supercritical pitchfork bifurcation. We have also shown the occurence of secondary instabilities like Eckhaus and zigzag instability.

Following *Knobloch and De Luca (1990)*, we derive in section 5, two nonlinear one-dimensional coupled Landau-Ginzburg type equations in complex amplitudes $A_R(X,\tau,T)$ and $A_L(X,\tau,T)$ of right-hand and left-hand travelling waves with complex coefficients near a supercritical Hopf bifurcation. Following *Matthews and Rucklidge (1993)*, we have dropped slow space dependence in X and obtained two coupled ordinary differential equations in $A_{1R}(T)$ and $A_{1L}(T)$ and discussed the stability regions of travelling and standing waves. We have shown the condition of Benjamin-Feir type instability for travelling and standing waves. In section 6, we write conclusions.

2. Basic equations

Consider a two-component electrically and thermally conducting fluid of infinite extent in the presence of a vertical magnetic field. Our intention in this paper is to examine thermal and compositional magneto-convection in the Earth's outer core. The fluid has density ρ which depends on both temperature T and concentration C of the lighter component fluid through the relation

$$\rho = \rho_0[1 - \alpha(T - T_0) - \beta(C - C_0)], \tag{2.1}$$

where $\alpha = -\rho_0^{-1}\partial\rho/\partial T$ and $\beta = -\rho_0^{-1}\partial\rho/\partial C$. Here α is the coefficient of thermal expansion and β is the parameter measuring the variation of the density with concentration, while T_0, C_0, ρ_0 are reference values. In the Earth's outer core both α and β are positive.

We use cartesian system of co-ordinates whose dimensionless vertical co-ordinate z and dimensionless horizontal co-ordinates x, y are scaled on d.

The velocity vector $\vec{V}(u,v,w)$, density ρ, temperature θ, concentration C, time t, pressure p and magnetic field $\vec{H}(H_x, H_y, H_z)$ are non-dimensionalized by scales κ/d, ρ_0, $\delta T = T - T_0$, $\delta C = C - C_0$, d^2/κ, $\rho_0\kappa^2/d^2$ and $\kappa H_0/\eta$. Here ν is viscosity, κ is thermal diffusivity, D_m is mass or material diffusivity, η is magnetic diffusivity, μ_m is magnetic permeability, g is acceleration due to gravity and κ_T is Soret coefficient (which measures the cross coupling between temperature gradients and mass fluxes and can have plus or minus sign but cannot be zero). The dimensionless parameters, $R, Q, \sigma_1, \sigma_2, \sigma_3, \sigma_4$ and ψ, required for the description of the motion are: thermal Rayleigh number, $R = \alpha g \Delta T d^3/\kappa\nu$, thermal Prandtl number, $Pr_1 \equiv \sigma_3^{-1} = \nu/\kappa$, Roberts number, $\sigma_2 = \kappa/\eta$, Schmidt number, $\sigma_4 = \kappa/D_m$ (we remind that Lewis number, $L = \sigma_4^{-1} = D_m/\kappa$), separation parameter, $\psi = -\kappa_T\beta/T_0\alpha$, and Chandrasekhar number, $Q = \mu_m H_0^2 d^2/4\pi\rho_0\nu\eta$. Finally, we formally introduce another "Prandtl number", $\sigma_1 = 1$, only due to the possibility to focus attention on some symmetries in the following relations. In Earth's outer core $L < 1$ and ψ can be positive or negative but not zero. The basic dimensionless equations for thermal and compositional magneto-convection in Earth's outer core in the Boussinesq approximation are

$$\nabla \cdot \vec{V} = 0, \quad \nabla \cdot \vec{H} = 0, \tag{2.2}$$

$$\begin{aligned} \sigma_3 \left[\partial_t \vec{V} + (\vec{V}\cdot\nabla)\vec{V}\right] - Q\sigma_2(\vec{H}\cdot\nabla)\vec{H} - Q\partial_z\vec{H} = \\ = -\nabla(P + \frac{Q\sigma_2}{2}|\vec{H}|^2) + R\,(\theta + \psi C)\,\hat{z} + \nabla^2\vec{V}, \end{aligned} \tag{2.3}$$

$$\sigma_1 \left[\partial_t \theta + (\vec{V}\cdot\nabla)\theta\right] = w\sigma_1 + \nabla^2\theta, \tag{2.4}$$

$$\sigma_4 \left[\partial_t C + (\vec{V}\cdot\nabla)C\right] = w\sigma_4 + \nabla^2 C - \nabla^2\theta, \tag{2.5}$$

$$\left[\sigma_2\partial_t - \nabla^2\right]\vec{H} = \nabla\times(\vec{V}\times\hat{z}) + \sigma_2\nabla\times(\vec{V}\times\vec{H}). \tag{2.6}$$

Geophysically acceptable velocities of propagating instabilities corresponding to geomagnetic secular variations occur only for Roberts number, $\sigma_2 = \kappa/\eta > 1$ (where instabilities develop in Ohmic diffusion time scale, d^2/η), $\sigma_2 = 2$ and 5, when the turbulence is present in the Earth's outer core). In the case of $\sigma_2 = \kappa/\eta \ll 1$ (based on molecular thermal diffusivity corresponding to the absence of turbulence in the Earth's outer core), the instabilities are extremely slow depending on the thermal diffusion time scale d^2/κ. Eqs. (2.2−2.6) can be reduced to a form

$$\mathcal{L}w = \mathcal{N}, \tag{2.7}$$

where

$$\mathcal{L} = \nabla^2\mathcal{D}_1\mathcal{D}_2\mathcal{D}_3\mathcal{D}_4 - R\mathcal{D}_2\partial_{xx}\left[\sigma_1\mathcal{D}_4 - \psi(\sigma_1\nabla^2 - \sigma_4\mathcal{D}_1)\right] - Q\mathcal{D}_1\mathcal{D}_4\partial_{zz}, \tag{2.8a}$$

$$\mathcal{N} = \mathcal{D}_1\mathcal{D}_2\mathcal{D}_4\, N - R\sigma_1\mathcal{D}_2\mathcal{D}_4\partial_{xx}(\vec{V}\cdot\nabla)\theta - R\psi\sigma_4\mathcal{D}_1\mathcal{D}_2\partial_{xx}(\vec{V}\cdot\nabla)C + \\ + R\sigma_1\psi\mathcal{D}_2\partial_{xx}\nabla^2(\vec{V}\cdot\nabla)\theta - Q\mathcal{D}_1\mathcal{D}_4\,\partial_z M \tag{2.8b}$$

$$\mathcal{D}_1 = \left(\sigma_1\partial_t - \nabla^2\right), \quad \mathcal{D}_2 = \left(\sigma_2\partial_t - \nabla^2\right),$$
$$\mathcal{D}_3 = \left(\sigma_3\partial_t - \nabla^2\right), \quad \mathcal{D}_4 = \left(\sigma_4\partial_t - \nabla^2\right),$$
$$N = \hat{z}\cdot\nabla\times\nabla\times\left[\sigma_3(\vec{V}\cdot\nabla)\vec{V} - Q\sigma_2(\vec{H}\cdot\nabla)\vec{H}\right],$$
$$\text{and} \quad M = \sigma_2\hat{z}\cdot\nabla\times\nabla\times\left[(\vec{H}\cdot\nabla)\vec{V} - (\vec{V}\cdot\nabla)\vec{H}\right].$$

The similarity of operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3,$ and $\mathcal{D}_4$ in Eq. (2.8) explains the reason to introduce $\sigma_1 = 1$ in the source equation (2.4).

Boundary conditions

We assume that fluid is confined between $z = 0$ and $z = 1$, where $z = 1$ corresponds to a mantle boundary. For perfectly conducting boundary with temperature and solute, we have

$$\theta = 0, C = 0 \text{ and } H_z = 0 \text{ on } z = 0, z = 1 \text{ for all } x, y.$$

Also the normal component of the velocity would vanish on $z = 0$, $z = 1$, i.e.

$$w = 0 \text{ on } z = 0, z = 1 \text{ for all } x, y.$$

Above mentioned boundary conditions have to be satisfied.

However, there are two more conditions to be imposed on velocity depending on the nature of the surface. In this paper we consider free surfaces or free-free boundary conditions, i.e., on surfaces the tangential stresses vanish ($P_{xz} = P_{yz} = 0$) which is equivalent to

$$P_{xz} = \mu\left(\partial_z u + \partial_x w\right) = 0,\ P_{yz} = \mu\left(\partial_z v + \partial_y w\right) = 0,$$

where $\mu = \nu\rho_0$ is dynamic viscosity. Since w vanishes for $x,\ y$ on $z = 0,\ z = 1$, it follows that $\partial_z u = \partial_z v = 0$ on a free surface $z = 0,\ z = 1$. Hence from equation of continuity, we have $\partial_{zz} w = 0$ on $z = 0,\ z = 1$ for all $x,\ y$. In this paper we have considered only the idealized stress-free conditions on the surface and vanishing of temperature fluctuations. Thus $W = D^2W = D^4W = 0$ at $z = 0,\ 1$. W and its even derivatives vanish at $z = 0$ and $z = 1$.

3. Linear stability analysis

We perform the linear stability analysis of the problem by substituting

$$w = W(z)e^{[i(q_x x + q_y y) + pt]} \tag{3.1}$$

into linearized version of Eq. (2.7) i.e., $\mathcal{L}w = 0$ and obtaining an equation

$$\begin{aligned}\mathcal{D}(\mathcal{D} - p\sigma_1)(\mathcal{D} - p\sigma_4)\left[(\mathcal{D} - p\sigma_2)(\mathcal{D} - p\sigma_3) - QD^2\right] w = \\ = -Rq^2(\mathcal{D} - p\sigma_2)\left[(\mathcal{D} - p\sigma_4) + \psi\left((1+\sigma_4)\mathcal{D} - p\sigma_4\right)\right] w,\end{aligned} \tag{3.2}$$

where $\mathcal{D} = D^2 - q^2$, $D = d/dz$ and $q^2 = q_x^2 + q_y^2$. In this paper we consider only the idealized boundary conditions. Hence w and all its even derivatives vanish at $z = 0$ and $z = 1$. Substituting $W = \sin \pi z$ and $p = i\omega$ (where ω stands for the frequency of oscillation) in Eq. (3.2), we get

$$R = \frac{K}{q^2}\left[\left(G_1\omega^6 + G_2\omega^4 - G_3\omega^2 + G_4\right) + i\omega\delta^4\left(A_1\omega^4 + A_2\omega^2 + A_3\right)\right] \tag{3.3}$$

where

$$\delta^2 = q^2 + \pi^2, \quad K = \frac{(\delta^4\psi_1^2 + \omega^2\psi_4^2)^{-1}}{(\delta^4 + \omega^2\sigma_2^2)},$$

$$\psi_1 = 1 + \psi(1+\sigma_4), \quad \psi_4 = (1+\psi)\sigma_4,$$

$$\psi_2 = \sigma_2 + \sigma_4 + \psi(\sigma_2 + \sigma_4 + \sigma_2\sigma_4),$$

$$G_1 = -\delta^2\sigma_2\psi_4 S_4,$$

$$G_2 = \delta^2\{\delta^4\left[\sigma_2\psi_4 S_2 - \psi_2 S_3\right] + Q_1\sigma_1\sigma_2\sigma_4\psi_4\},$$

$$G_3 = \delta^6\{\delta^4\left[\sigma_2\psi_4 - \psi_2 S_1 + S_2\psi_1\right] + Q_1\left[\sigma_2\psi_4 - (\sigma_1+\sigma_4)\psi_2 + \sigma_1\sigma_4\psi_1\right]\},$$

$$G_4 = \delta^{10}\psi_1(\delta^4 + Q_1),$$

$$A_1 = \sigma_2\psi_4 S_3 - S_4\psi_2,$$

$$A_2 = \delta^4\left[S_2\psi_2 - \sigma_2\psi_4 S_1\right] - Q_1\left[\sigma_1\sigma_4\psi_2 - \sigma_2\psi_4(\sigma_1+\sigma_4)\right],$$

$$A_3 = \delta^4\{\delta^4\left[\psi_1 S_1 - \psi_2\right] + Q_1\left[\psi_1(\sigma_1+\sigma_4) - \psi_2\right]\}.$$

Here using $Q_1 = Q\pi^2$ for brevity and also symmetric polynomials $S_1 = \sigma_1+\sigma_2+\sigma_3+\sigma_4$, $S_2 = \sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4$, $S_3 = \sigma_1\sigma_2\sigma_3+\sigma_1\sigma_2\sigma_4+\sigma_1\sigma_3\sigma_4+\sigma_2\sigma_3\sigma_4$, and $S_4 = \sigma_1\sigma_2\sigma_3\sigma_4$ of Prandtl numbers $\sigma_1, \sigma_2, \sigma_3$, and σ_4.

3.1. Stationary convection ($\omega = 0$)

Substituting $\omega = 0$ in to Eq. (3.2), we get

$$R_s = \frac{\delta^2(\delta^4 + Q\pi^2)}{\psi_1 q^2} \tag{3.4}$$

Here R_s is the value of R for the stationary convection. The minimum value of R_s obtained for $q = q_{sc}$ where

$$2\left(\frac{q_{sc}}{\pi}\right)^6 + 3\left(\frac{q_{sc}}{\pi}\right)^2 = 1 + \frac{Q}{\pi^2}. \tag{3.5}$$

The wave number is identical to that for the single component fluid, while the threshold for the onset of stationary convection at pitchfork bifurcation is given by Eq. (3.6) with $q = q_{sc}$. Thus

$$R_{sc} = \frac{\delta_{sc}^2(\delta_{sc}^4 + Q\pi^2)}{\psi_1 q_{sc}^2}. \tag{3.6}$$

3.2. Oscillatory convection $\left(\omega^2 > 0\right)$

For oscillatory convection $\omega \neq 0$ and from Eq. (3.2), R will be complex. But the physical meaning of R requires it to be real. The condition that R is real implies that imaginary part of Eq. (3.2) is zero, i.e.,

$$\omega^4 A_1 + \omega^2 A_2 + A_3 = 0. \tag{3.7}$$

Substituting $W = \sin \pi z$ in Eq. (3.2), we get a fourth degree polynomial equation in p of the form

$$a_4 p^4 + a_3 p^3 + a_2 p^2 + a_1 p + a_0 = 0, \tag{3.8}$$

where

$$\begin{aligned}
a_4 &= \delta^2 S_4, \\
a_3 &= \delta^4 S_3, \\
a_2 &= \delta^6 S_2 - Rq^2 \psi_4 \sigma_2 + \delta^2 Q_1 \sigma_1 \sigma_4, \\
a_1 &= \delta^2 [\delta^6 S_1 - Rq^2 \psi_2 + \delta^2 Q_1 (\sigma_1 + \sigma_4)], \\
a_0 &= \delta^4 (\delta^6 - Rq^2 \psi_1 + \delta^2 Q_1).
\end{aligned} \tag{3.9}$$

Setting $p = i\omega$ in Eq. (3.8) and equating its real and imaginary parts to zero, we get

$$a_4 \omega^4 - a_2 \omega^2 + a_0 = 0, \tag{3.10}$$

$$a_3 \omega^2 - a_1 = 0. \tag{3.11}$$

From Eq. (3.8), if $\omega = 0$ then $a_0 = 0$ and we get stationary convection and R_s is determined by putting $R = R_s$ in $a_0 = 0$. Thus $\omega = 0$ and $a_0 = 0$ are the conditions for the pitchfork bifurcation corresponding to stationary convection. From Eq. (3.10), we can have marginal stability if $\omega^2 = a_1/a_3 \, (a_1 > 0)$ and

$$a_4 a_1^2 - a_1 a_2 a_3 + a_0 a_3^2 = 0. \tag{3.12}$$

In this case we get oscillatory convection and R_o (the value of R for the oscillatory convection) is obtained by putting $R = R_o$ in the expressions a_0, a_1, a_2, a_3, a_4 of the set of Eqs. (3.9) into Eq. (3.12). Thus we get a quadratic equation in R_o. The codimension-two point is determined by the intersection of two lines $a_0 = 0$ and $a_1a_4 - a_2a_3 = 0$ under the condition $a_1 > 0$ in (ψ, R)-space. This corresponds to the simultaneous occurrence of pitchfork and Hopf bifurcation and quasiperiodic solutions of the system can be obtained in the nonlinear regime.

Takens-Bogdanov bifurcation point is determined by the intersection of the two curves $a_0 = 0$ and $a_1 = 0$ in (ψ, R)-space. Thus Takens-Bogdanov bifurcation point corresponds to a double zero eigenvalue of the linear growth rate. At the codimension-two point, we have

$$R_{sc}(q_{sc}) = R_{oc}(q_{oc}) \text{ but } q_{sc} \neq q_{oc}, \tag{3.13}$$

and at the Takens-Bogdanov bifurcation point, we have

$$R_s(q_s) = R_o(q_o) = R_c(q_c) \text{ and } q_s = q_o = q_c. \tag{3.14}$$

Eliminating R from $a_0 = a_1 = 0$, we get Takens-Bogdanov bifurcation point at

$$\psi = \psi^* = \frac{\delta^4 S_1 + Q_1(\sigma_1 + \sigma_4) - (\delta^4 + Q_1)(\sigma_2 + \sigma_4)}{(\delta^4 + Q_1)[\sigma_2 + \sigma_4 + \sigma_2\sigma_4] - [\delta^4 S_1 + Q_1(\sigma_1 + \sigma_4)](1 + \sigma_4)}, \tag{3.15}$$

From Eq. (3.15), ψ^* is always negative if $\sigma_1 \geq 1$. The codimension-two point is an intersection between Hopf bifurcation and pitchfork bifurcation with distinct wave numbers in (ψ, R) plane. At Takens-Bogdanov bifurcation, the Hopf bifurcation and pitchfork bifurcation neutral curves intersect, and only a single wave-number is present. Thus at a Takens-Bogdanov bifurcation point the oscillatory neutral curve intersects the stationary convection curve and the frequency on the oscillatory neutral curve approaches zero as the intersection point is approached. In Figs. 1-2, each solid line stands for stationary convection (pitchfork bifurcation) and dotted line stands for oscillatory convection (Hopf bifurcation).

In these Figs. 1-2, we have showed the effect of several physical parameters, like $Q, \sigma_2, \sigma_3, \sigma_4$ and ψ on the onset of both stationary convection and oscillatory convection. When a physical parameter increases for the remaining fixed parameters, the onset of instabilities increases i.e., the onset of stationary convection and oscillatory convection inhibit when a parameter increases with the remaining fixed parameters. We have $\psi = \psi^*$ at the

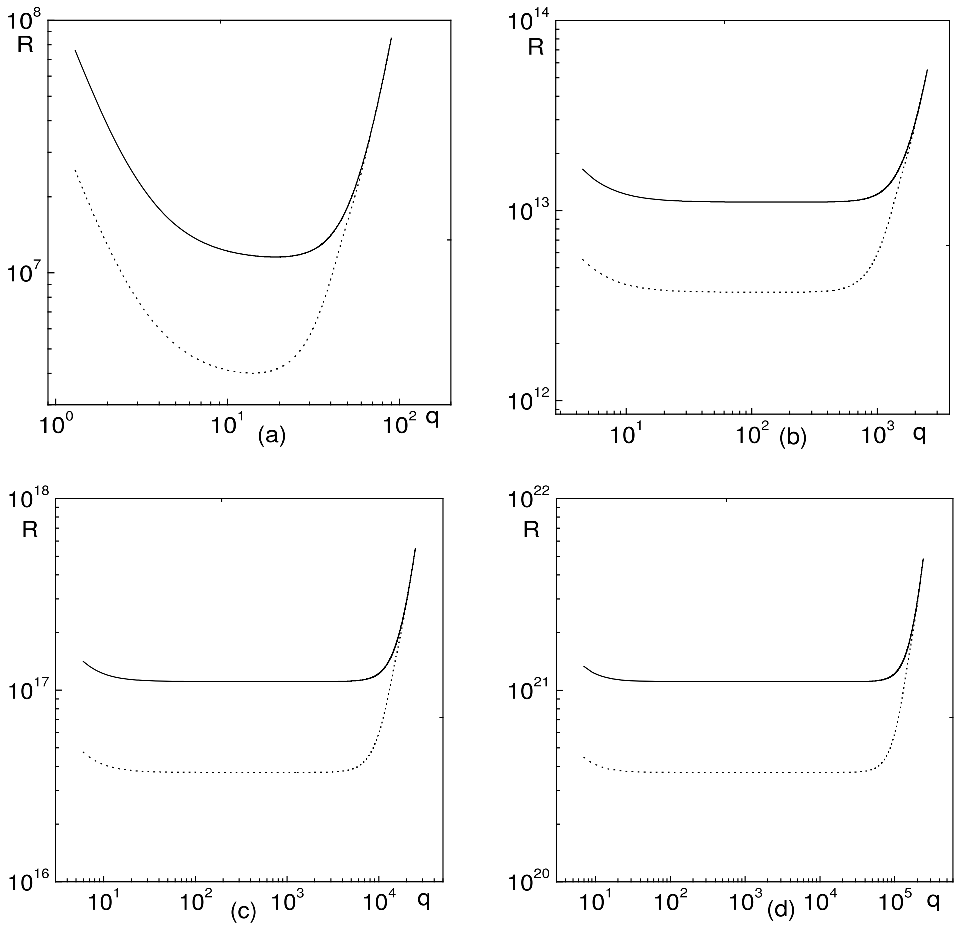

Fig. 1. Numerically calculated marginal stability curves are plotted in $(R, q)-$ plane for $\sigma_2 = 2$, $L = 0.1$, $\psi = -0.01$ and (a) $Q = 10^6$ (b) $Q = 10^{12}$ (c) $Q = 10^{16}$, (d) $Q = 10^{20}$, then the onset of stationary convection and the onset of oscillatory convection increases (stationary convection stands for solid lines and oscillatory convection stands dotted lines).

Takens-Bogdanov bifurcation point. In the limit $\psi \longrightarrow \psi^*$, the frequency of the oscillatory instability tends to zero. At codimension two bifurcation point let $\psi = \psi'$ for a Chandrashekar number Q. If $\psi < \psi'$, we get first

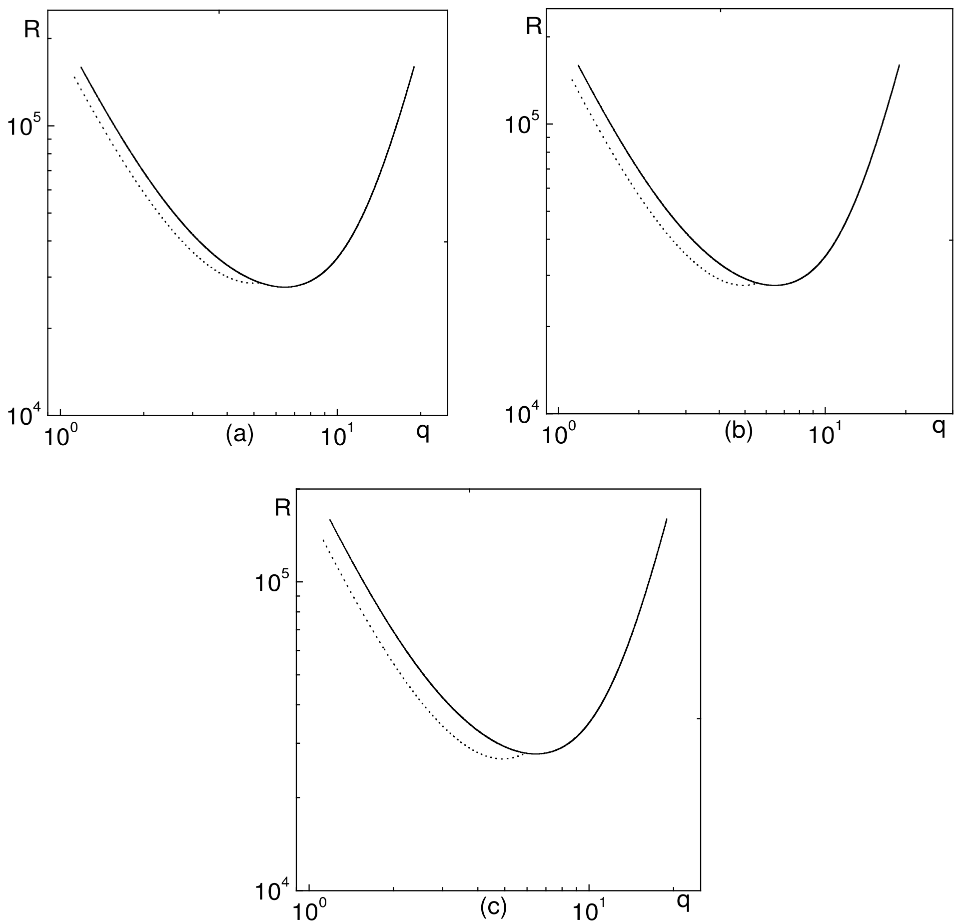

Fig. 2. Neutral curves for the stationary bifurcation (solid lines) and for the Hopf bifurcation (dashed lines) near the codimension two point for $Q = 2000$, $\psi = -0.0001$, $L = 0.1, (a)$ $\sigma_2 = 1.14$, (b) $\sigma_{2c} = 1.167$, (c) $\sigma_2 = 1.2$. $x-$ axis wave number, $y-$ Rayleigh numbers R_s, R_o.

instability as oscillatory convection. If $\psi > \psi'$, then we get stationary convection as a first instability. For $a_0 = a_1 = a_2 = 0$, which gives $\psi = \psi^{**}$, corresponding to codimension-three bifurcation point and $\omega = 0$ is a triple zero eigenvalue. Eliminating Q and R from $a_0 = a_1 = a_2 = 0$, we get

$$\psi = \psi^{**} =$$
$$= \frac{[S_1\sigma_4 - S_2 + \sigma_1\sigma_4](\sigma_2 - \sigma_1)}{S_1\sigma_4[\sigma_1(\sigma_1+\sigma_4) - \sigma_2] - S_2[\sigma_2(\sigma_1+\sigma_4)+\sigma_1] - \sigma_1\sigma_4[\sigma_2(\sigma_1+\sigma_4)+\sigma_4]}, \tag{3.16}$$

4. Two dimensional Landau-Ginzburg equation at the onset of stationary convection

The existence of a threshold (critical value of the Rayleigh number, $R = R_{sc}$) and the cellular structure (critical wave number, $q = q_{sc}$) for a fixed Lewis number L and separation parameter ψ are main characteristics of the stationary convection due to compositional and thermal buoyancy. In this section, we consider the region near the onset of stationary convection by introducing ϵ as

$$\epsilon^2 = \frac{(R_s - R_{sc})}{R_{sc}} \ll 1. \tag{4.1}$$

To simplify the problem we assume the formation of rolls parallel to the y-axis, so that y-dependence disappears from Eq. (2.7). The z-dependence is contained entirely in the sine and cosine functions which ensures that free-free boundary conditions are satisfied. For values of the control parameter $R = R_s$ close to the threshold value $R_{sc}\,(\epsilon^2 \ll 1)$, we assume solutions of Eqs. (2.2–2.6) in powers of ϵ as :

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + ..., \tag{4.2}$$

where $f = (u, v, w, \theta, c, H_x, H_y, H_z)$ with the first approximation is given by the eigenvector of the linearized problem:

$$w_0 = A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c.c.,$$
$$u_0 = \frac{i\pi}{q_{sc}}\left[A(X,Y,T)e^{iq_{sc}x}\cos\pi z - c.c.\right],$$
$$v_0 = 0, H_{y_0} = 0,$$

$$H_{x_0} = -\frac{i\pi^2}{\delta_{sc}^2 q_{sc}} \left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z - c.c.\right],$$
$$H_{z_0} = \frac{\pi}{\delta_{sc}^2} \left[A(X,Y,T)e^{iq_{sc}x}\cos\pi z + c.c.\right],$$
$$\theta_0 = \frac{1}{\delta_{sc}^2} \left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c.c.\right],$$
$$C_0 = \frac{(1+\sigma_4)}{\delta_{sc}^2} \left[A(X,Y,T)e^{iq_{sc}x}\sin\pi z + c.c.\right], \tag{4.3}$$

where $\delta_{sc}^2 = \pi^2 + q_{sc}^2$. Here $c.c.$ stands for complex conjugate, $e^{iq_{sc}x}\sin\pi z$ is the critical mode for the linear problem at $R = R_{sc}$ and $q = q_{sc}$. The complex amplitude $A(X,Y,T)$ depends on the slow variables. The independent variables $x,\ y,\ z,\ t$ are scaled by introducing multiple scales

$$X = \epsilon x,\ Y = \epsilon^{\frac{1}{2}} y,\ Z = z,\ T = \epsilon^2 t, \tag{4.4}$$

and these formally separate the fast and slow dependent variables in f. It should be noted that the difference in scaling in the two directions reflects the inherent symmetry breaking of instability which was chosen here with wave vector in x-direction.

The differential operators can be expressed as

$$\partial_x \longrightarrow \partial_x + \epsilon\partial_X,\quad \partial_y \longrightarrow \epsilon^{\frac{1}{2}}\partial_Y,$$
$$\partial_z \longrightarrow \partial_Z,\quad \partial_t \longrightarrow \epsilon^2\partial_T. \tag{4.5}$$

Using (4.5), the operator (2.8a) and (2.8b) can be written as

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \cdots,$$
$$\mathcal{N} = \epsilon^2\mathcal{N}_0 + \epsilon^3\mathcal{N}_1 + \cdots, \tag{4.6}$$

where

$$\mathcal{L}_0 = \nabla^4\left[\nabla^6 - R_{sc}\psi_1\partial_{xx} - Q\nabla^2\partial_{zz}\right], \tag{4.7}$$
$$\mathcal{L}_1 = (2\partial_{xX} + \partial_{YY})\left[5\nabla^6 - 2R_{sc}\psi_1\partial_{xx} - 3Q\nabla^2\partial_{zz}\right]\nabla^2, \tag{4.8}$$
$$\mathcal{L}_2 = \partial_T\left[S_1\nabla^6 - R_{sc}\psi_2\partial_{xx} - Q(\sigma_1+\sigma_4)\nabla^2\partial_{zz}\right]\nabla^2 - \\ - \partial_{XX}\left[5\nabla^6 - 2R_{sc}\psi_1\partial_{xx} - 3Q\nabla^2\partial_{zz}\right]\nabla^2 - \\ - 2\left(2\partial_{xX} + \partial_{YY}\right)\left[10\nabla^6 - R_{sc}\psi_1\partial_{xx} - 3Q\nabla^2\partial_{zz}\right]. \tag{4.9}$$

Using (4.1−4.5) into Eq. (2.7), and using the definitions of $\mathcal{L}$ and $\mathcal{N}$ from (4.6), we get equating coefficients of various powers of ϵ to zero

$$\mathcal{L}_0 w_0 = 0, \tag{4.10}$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \tag{4.11}$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \tag{4.12}$$

Substituting the value of w_0 from (4.3) into (4.10) and using (4.7), we get

$$R_{sc} = \frac{\delta_{sc}^2(\delta_{sc}^4 + Q_1)}{\psi_1 q_{sc}^2}. \tag{4.13}$$

Substituting the value of w_0 into $\mathcal{L}_1 w_0 = 0$, we get critical wave number. In Eq. (4.11), $\mathcal{N}_0 = 0$, $\mathcal{L}_1 w_0 = 0$ implies that Eq. (4.11) reduces to $w_1 = 0$. Similarly $u_1 = 0, v_1 = 0, H_{x_1} = 0, H_{y_1} = 0$

$$\theta_1 = -\frac{1}{2\pi\delta_{sc}^2} |A|^2 \sin 2\pi z, \tag{4.14}$$

$$C_1 = -\frac{1}{2\pi\delta_{sc}^2}(1 + \sigma_4 + \sigma_4^2) |A|^2 \sin 2\pi z, \tag{4.15}$$

$$H_{z_1} = -\frac{\sigma_2\pi^2}{2\delta_{sc}^2 q_{sc}^2} \left[|A|^2 e^{2iq_{sc}x} + c.c\right] \cos 2\pi z. \tag{4.16}$$

Substituting zero order and first order solution in (2.7) and equating coefficients of $\sin \pi z$ in $\mathcal{N}_1 - \mathcal{L}_2 w_0$ to zero, we get

$$\lambda_0 \partial_T A - \lambda_1 (\partial_X - \frac{i}{2q_{sc}} \partial_{YY})^2 A - \lambda_2 A + \lambda_3 |A|^2 A = 0, \tag{4.17}$$

where

$$\begin{aligned}
\lambda_0 &= -\delta_{sc}^8 S_1 + R_{sc} q_{sc}^2 \delta_{sc}^2 \psi_\sigma - Q_1(\sigma_1 + \sigma_4)\delta_{sc}^4, \\
\lambda_1 &= 4q_{sc}^2 \left[10\delta_{sc}^6 - 3Q_1\delta_{sc}^2 + R_{sc} q_{sc}^2 \psi_1\right], \\
\lambda_2 &= R_{sc} q_{sc}^2 \delta_{sc}^4 \psi_1, \\
\lambda_3 &= \frac{R_{sc} q_{sc}^2 \delta_{sc}^2}{2} \left[1 + \psi(\sigma_1 + \sigma_4)(1 + \sigma_4^{-2})\right] + \frac{Q_1 \pi^2 \delta_{sc}^4 \sigma_2^2}{2q_{sc}^2}.
\end{aligned} \tag{4.18}$$

Eq. (4.17) is called Landau-Ginzburg equation. Eq. (4.17) is meaningful only if λ_0, λ_2 and λ_3 are positive. For $\lambda_3 > 0$, we get a forward bifurcation (supercritical pitchfork bifurcation). Landau-Ginzburg equation is valid only for $\lambda_3 > 0$. If $\lambda_3 < 0$ the bifurcation is subcritical (Fig. 3).

At $\lambda_3 = 0$, we get tricritical bifurcation point. Dropping the time-dependent term from Eq. (4.17), we get

$$\frac{d^2A}{dX^2} + \frac{\lambda_2}{\lambda_1}\left(1 - \frac{\lambda_3}{\lambda_2}|A|^2\right)A = 0. \tag{4.19}$$

The solution of Eq. (4.19) is given by

$$A(X) = A_0 \tanh\left(X/\Lambda\right), \tag{4.20}$$

where

$$A_0 = (\lambda_2/\lambda_3)^{\frac{1}{2}} \text{ and } \Lambda = (2\lambda_1/\lambda_2)^{\frac{1}{2}}. \tag{4.21}$$

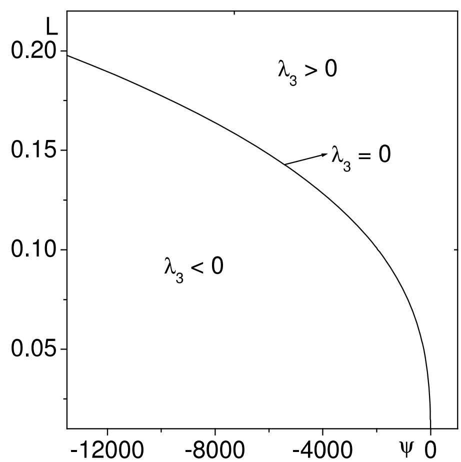

Fig. 3. Above figure is plotted for $Q = 1000$, $\sigma_2 = 2$. λ_3 is the nonlinear coefficient of Landau-Ginzburg equation at the onset of stationary convection. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$ and subcritical if $\lambda_3 < 0$.

4.1. Long wave-length instabilities (secondary instabilities)

The two-dimensional Landau-Ginzburg Eq. (4.17), can be written in fast variables x, y, t and $A(X,Y,T) = A(x,y,t)/\epsilon$, as

$$\lambda_0\partial_t A - \lambda_1(\partial_x - \frac{i}{2q_{sc}}\partial_{yy})^2 A - \epsilon^2\lambda_2 A + \lambda_3|A|^2 A = 0. \tag{4.22}$$

In order to study the properties of a structure with a given phase winding number δk, we substitute

$$A\left(x,y,t\right)=A_{1}\left(x,y,t\right)e^{i\delta kx}, \tag{4.23}$$

into the Eq. (4.22) and we obtain

$$\lambda_0\partial_t A_1=\left(\epsilon^2\lambda_2-\lambda_1(\delta k)^2\right)A_1+2i\lambda_1\delta k(\partial_x-\frac{i}{2q_{sc}}\partial_{yy})A_1+ \\ +\lambda_1(\partial_x-\frac{i}{2q_{sc}}\partial_{yy})^2A_1-\lambda_3|A_1|^2A_1=0. \tag{4.24}$$

The steady state uniform solution of Eq. (4.24) is

$$A_1=A_{10}=[\left(\epsilon^2\lambda_2-\lambda_1(\delta k)^2\right)\lambda_3^{-1}]^{\frac{1}{2}}. \tag{4.25}$$

Let $\tilde{u}(x,y,t)+i\tilde{v}(x,y,t)$ be an infinitesimal perturbation from a uniform steady state solution A_{10} given by Eq. (4.25). Now substituting

$$A_1=A_{10}=[\left(\epsilon^2\lambda_2-\lambda_1(\delta k)^2\right)\lambda_3^{-1}]^{\frac{1}{2}}+\tilde{u}+i\tilde{v},$$

into Eq. (4.24) and equating real and imaginary parts, we obtain

$$\lambda_0\partial_t\tilde{u}=[-2\left(\epsilon^2\lambda_2-\lambda_1(\delta k)\right)^2+ \\ +\lambda_1(\partial_{xx}+\frac{\delta k\partial_{yy}}{q_{sc}}-\frac{\partial_{yyyy}}{4q_{sc}^2})]\tilde{u}-(2\lambda_1\delta k-\frac{\lambda_1\partial_{yy}}{q_{sc}})\partial_x\tilde{v}, \tag{4.26a}$$

$$\lambda_0\partial_t\tilde{v}=(2\lambda_1\delta k-\frac{\lambda_1}{q_{sc}}\partial_{yy})\partial_x\tilde{u}+\lambda_1(\partial_{xx}+\frac{\delta k}{q_{sc}}\partial_{yy}-\frac{1}{4q_{sc}^2}\partial_{yyyy})\tilde{v}. \tag{4.26b}$$

We analyze Eqs. (4.26a) and (4.26b) by using normal modes of the form

$$\tilde{u}=Ue^{St}\cos\left(q_xx\right)\cos\left(q_yy\right),\tilde{v}=Ve^{St}\sin\left(q_xx\right)\cos\left(q_yy\right). \tag{4.27}$$

Putting Eq. (4.27) in Eqs. (4.26a) and (4.26b) we get,

$$\left[\lambda_0S+2\left(\epsilon^2\lambda_2-\lambda_1(\delta k)^2\right)+\chi_1\right]U+\lambda_1q_x\chi_2V=0, \tag{4.28a}$$

$$\lambda_1q_x\chi_2U+\left(\lambda_0S+\chi_1\right)V=0. \tag{4.28b}$$

Here $\chi_1 = \lambda_1[q_x^2 + (q_y^2 \delta k)/q_{sc} + q_y^4/4q_{sc}^2]$, $\chi_2 = (2\delta k + q_y^2/q_{sc})$. On solving Eq. (4.28a) and Eq. (4.28b) we get,

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\chi_1\right] + \\ + \left[2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \chi_1\right]\psi_1 - q_x^2\lambda_1\chi_2 = 0,$$

whose roots $(S\pm)$ are real. Here $(S\pm)$ is defined as

$$(S\pm) = -\frac{1}{\lambda_0^2}\left\{\left(2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\chi_1\right) \pm \\ \pm\left(2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right)^2 + \lambda_1^2 q_x^2 \chi_2^2\right)^{\frac{1}{2}}\right\}. \tag{4.29}$$

Solution $S(-)$ is clearly negative, thus the corresponding mode is stable and if $S(+)$ is positive then rolls can be unstable. Symmetry considerations help us to restrict the study of $S(+)$ to a domain $q_x \geq 0, q_y \geq 0$.

4.1.1. Longitudinal perturbations and Eckhaus instability

Inserting $q_y = 0$ into Eq. (4.29), we get

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \\ + \lambda_0\lambda_1 q_x^2\right] + \lambda_1 q_x^2\left[2\left(\epsilon^2\lambda_2 - 3\lambda_1(\delta k)^2\right) + q_x^2\right] = 0,$$

since the roots are real and their sum always negative, the pattern is stable as long as both roots are negative, i.e., their product is positive. The cell pattern becomes unstable when the product is negative, i.e., when

$$q_x^2 \leq 2\left(3\lambda_1\delta k^2 - \epsilon^2\lambda_2\right),$$

for this requires $|\delta k| \geq \sqrt{(\epsilon^2\lambda_2/3\lambda_1)}$, this condition defines the domain of Eckhaus instability. The above condition implies that the most unstable wave vector tends to zero, when $|\delta k| \to \sqrt{(\epsilon^2\lambda_2/3\lambda_1)}$.

4.1.2. Transverse perturbations and zigzag instability

Let us consider $q_x = 0$ into Eq. (4.29), we get

$$\lambda_0^2 S^2 + 2S\left[2\lambda_0\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \lambda_0\chi_1^y\right] + \left[2\left(\epsilon^2\lambda_2 - \lambda_1(\delta k)^2\right) + \chi_1^y\right]\chi_1^y = \\ = 0,$$

where $\chi_1^y = \lambda_1 \left(q_y^2 \delta k / q_{sc} + q_y^4 / 4 q_{sc}^2 \right)$. The two eigenmodes are uncoupled and we have $S(-)$,

$$S(-) = -2 \left(\epsilon^2 \lambda_2 - \lambda_1 (\delta k)^2 \right) - \frac{\lambda_1}{q_{sc}} \delta k q_y^2 - \frac{\lambda_1}{4 q_{sc}^2} q_y^2 < 0,$$

for one of them. The other is amplified when

$$S(+) = -\lambda_1 q_y^2 \left(\delta k + \frac{q_y^2}{4 q_{sc}} \right) > 0.$$

This implies that $\delta k < 0$, the above condition defines the domain of the zigzag instability. When $\delta k \to 0$ from below the wave vector q_y of the instability also tends to zero, while the growth rate varies as q_y^2. We have studied the effect of magnetic field on long wave length instabilities. We have observed that Eckhaus instability and zigzag instability regions increases when Q increases (see Fig. 4).

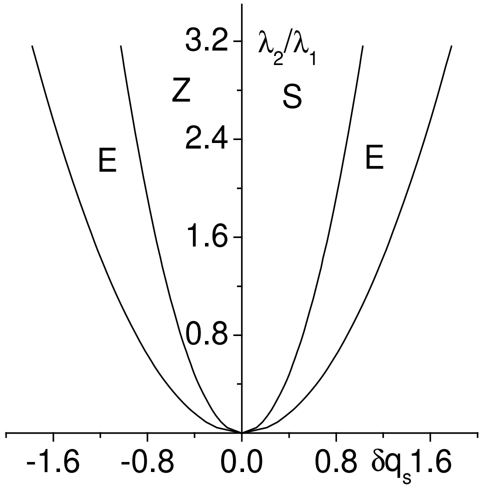

Fig. 4. Numerically computed secondary instability regions of Eckhaus instability (E), zigzag instability (Z) and stable region (S) are plotted in $(\lambda_2/\lambda_1, \delta q_s)-$ plane for $L = 0.1$, $\psi = -0.01, \sigma_2 = 2, Q = 2000$. As $|\delta q_s|$ increases then the secondary instability regions increases.

4.2. Heat transport by convection

The maximum of steady amplitude A is denoted by $|A_{max}|$ which is given as

$$|A_{max}| = \left(\epsilon^2 \lambda_2 \lambda_3^{-1}\right)^{\frac{1}{2}} . \tag{4.30}$$

Eq. (4.30), is obtained either from Eq. (4.20) with $\tanh(X/\Lambda) = 1$ or from Eq. (4.23), with $\delta q_s = 0$ and $A_1 = A_{10}$. We use $|A_{max}|$ to calculate Nusselt number Nu.

To discuss the heat tranfer near the neutral region, we express it through the Nusselt number defined as $Nu = Hd/\kappa\Delta T$, which is the ratio of the heat transported across any layer to the heat which would be transported by conduction alone. Here H is the rate of heat transfer per unit area and is defined as

$$H = -\langle \frac{\partial T_{total}}{\partial z'} \rangle_{z'=0}. \tag{4.31}$$

In (4.31), angular brackets correspond to a horizontal average.

The Nusselt number can be calculated in terms of amplitude A and it is given as

$$Nu = 1 + \frac{\epsilon^2}{\delta_{sc}^2} |A_{max}|^2. \tag{4.32}$$

From Eq. (4.32), we get conduction for $R \leq R_{sc}$ and convection for $R > R_{sc}$. Since the amplitude equation is valid for $\lambda_3 > 0$, this is possible for $R > R_{sc}$ (supercritical). Thus we get $Nu > 1$ for $R > R_{sc}$. We get convection for $Nu > 1$ and conduction for $Nu \leq 1$. In stationary convection Nu increase implies the increase of heat transported by steady mode convection.

In the problem of double diffusive convection with magnetic field, Nu depends on $\psi, \sigma_2, \sigma_3, \sigma_4,$ and Q. We have computed Nu for different values of Q, for some fixed values of other parameters and observed that Nu increases as Q decreases (see Figs. 5(a) and 5(b)). This implies that magnetic field inhibits the heat transport. The parameters L, σ_1 and σ_2 show the same result as Q shows on Nu. When the other parameter $\psi < 0$, decreases then the Nusselt number decreases.

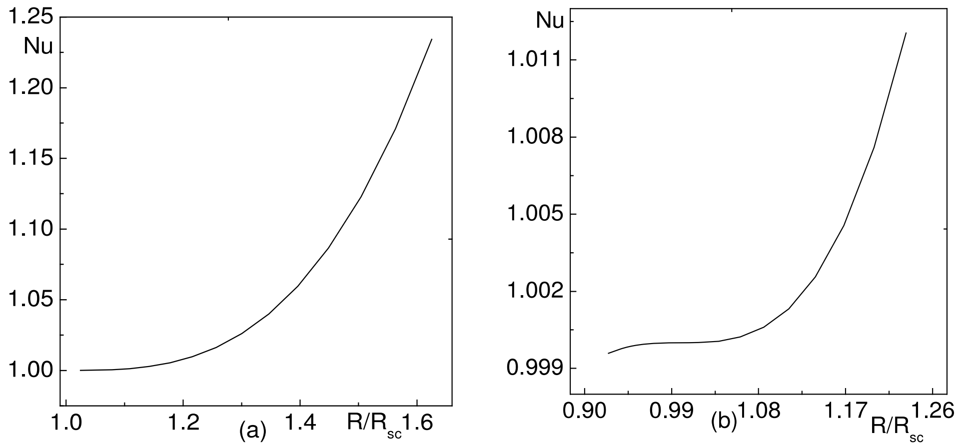

Fig. 5. Graph (a) is plotted for $Q = 1000$ and Graph (b) is plotted for $Q = 2000$ for the fixed values of $L = 0.1$, $\psi = 0.001$, $\sigma_2 = 2$ in $(Nu, R/R_{sc})-$ plane. In Graphs (a) and (b), as R/R_{sc} increases then Nu (Nusselt number) increases.

5. Oscillatory convection at the supercritical Hopf bifurcation

The existence of a threshold (critical value of Rayleigh number for the onset of oscillatory convection $R = R_{oc}$) and a cellular structure (critical wave number $q = q_{oc}$) and ψ are main characteristics of the oscillatory convection. In this section we treat region near the onset of oscillatory convection. Here the axis of the cylindrical rolls is taken as y-axis, so that y-dependence disappears from equation $\mathcal{L}w = \mathcal{N}$. The z-dependence contained entirely in sin and cos functions which ensure that the free-free boundary conditions are satisfied.

The purpose of this section is to derive coupled one dimensional nonlinear time dependent Landau-Ginzburg type equations near the onset of oscillatory convection at supercritical Hopf bifurcation. We introduce ϵ as

$$\epsilon^2 = \frac{(R_o - R_{oc})}{R_{oc}} \ll 1. \tag{5.1}$$

We assume that

$$w_0 = \left[A_{1L} e^{i(q_{oc}x + \omega_{oc}t)} + A_{1R} e^{i(q_{oc}x - \omega_{oc}t)} + c.c.\right] \sin \pi z,$$

is a solution to linearized equation $\mathcal{L}w = 0$, which satisfies free-free boundary conditions. Here A_{1L} denotes the amplitude of left travelling wave of the roll and A_{1R} denotes the amplitude of right travelling wave of the roll, which depends on slow space and time variables (*Knobloch and Luca, 1990*)

$$X = \epsilon x, \ \tau = \epsilon t, \ T = \epsilon^2 t, \tag{5.2}$$

and assume that $A_{1L} = A_{1L}(X, \tau, T)$, $A_{1R} = A_{1R}(X, \tau, T)$. The differential operators can be expressed as

$$\partial_x \ \longrightarrow \ \partial_x + \epsilon \partial_X, \quad \partial_t \ \longrightarrow \ \partial_t + \epsilon \partial_\tau + \epsilon^2 \partial_T. \tag{5.3}$$

The solution of basic equations can be sought as power series in ϵ,

$$f = \epsilon f_0 + \epsilon^2 f_1 + \epsilon^3 f_2 + \cdots , \tag{5.4}$$

where $f = (u, v, w, \theta, c, H_x, H_y, H_z)$ with the first approximation is given by eigenvector of the linearized problem:

$$\begin{aligned}
u_0 &= \frac{i\pi}{q_{oc}} \left[A_L(X,\tau,T) e^{i(q_{oc}x + \omega_{oc}t)} + A_R(X,\tau,T) e^{i(q_{oc}x - \omega_{oc}t)} - c.c. \right] \cos \pi z, \\
v_0 &= 0, H_{y_0} = 0, \\
\theta_0 &= \left[\frac{1}{e_1} A_L(X,\tau,T) e^{i(q_{oc}x + \omega_{oc}t)} + \frac{1}{e_1^*} A_R(X,\tau,T) e^{i(q_{oc}x - \omega_{oc}t)} + c.c. \right] \sin \pi z, \\
C_0 &= \left[h_1 A_L(X,\tau,T) e^{i(q_{oc}x + \omega_{oc}t)} + h_1^* A_R(X,\tau,T) e^{i(q_{oc}x - \omega_{oc}t)} + c.c. \right] \sin \pi z, \\
H_{x_0} &= -\frac{i\pi^2}{q_{sc}} \left[\frac{1}{e_2} A_L(X,\tau,T) e^{i(q_{oc}x + \omega_{oc}t)} + \right. \\
&\qquad \left. + \frac{1}{e_2^*} A_R(X,\tau,T) e^{i(q_{oc}x - \omega_{oc}t)} - c.c. \right] \sin \pi z, \\
H_{z_0} &= \pi \left[\frac{1}{e_2} A_L(X,\tau,T) e^{i(q_{oc}x + \omega_{oc}t)} + \right. \\
&\qquad \left. + \frac{1}{e_2^*} A_R(X,\tau,T) e^{i(q_{oc}x - \omega_{oc}t)} + c.c. \right] \cos \pi z,
\end{aligned} \tag{5.5}$$

where and henceforth, $\delta_{oc}^2 = (\pi^2 + q_{oc}^2)$, $e_j = (\delta_{oc}^2 + i\omega_{oc}\sigma_j)$, $j = 1, 2, 3, 4$; $h_1 = e_4^{-1}(\sigma_4 + e_1^{-1}\delta_{oc}^2)$, and $A_L = A_{1L} + \epsilon A_{2L} + \ldots$, $A_R = A_{1R} + \epsilon A_{2R} + \ldots$ The e_j^* and h_1^* are complex conjugates of e_j and h_1 for $j = 1, 2, 3, 4$.

We expand the linear operator $\mathcal{L}$ and nonlinear term $\mathcal{N}$ as the following power series

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 + \cdots \tag{5.6a}$$

$$\mathcal{N} = \epsilon^2\mathcal{N}_0 + \epsilon^3\mathcal{N}_1 + \cdots . \tag{5.6b}$$

Substituting Eqs. (4.2) and (5.3) into $\mathcal{L}w = \mathcal{N}$, for each order of ϵ, we get

$$\mathcal{L}_0 w_0 = 0, \tag{5.7a}$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \tag{5.7b}$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \tag{5.7c}$$

Here

$$\begin{aligned}
\mathcal{L}_0 &= E_4\nabla^2 - R_{oc}e_2\partial_{xx}\mathcal{D}_\psi - Qe_1e_4\nabla^2\partial_{zz},\\
\mathcal{L}_1 &= \partial_\tau\mathcal{F}_1 + 2\partial_{xX}\mathcal{F}_2,\\
\mathcal{L}_2 &= \partial_T\mathcal{F}_1 + 4\partial_{xxXX}\left\{E_2\nabla^2 - E_3 + Q(e_4+e_1-\nabla^2)\partial_{zz}+\right.\\
&\quad \left.+R_{oc}(\psi_1(e_2-\partial_{xx}) + \mathcal{D}_\psi) + \partial_{XX}\mathcal{F}_2\right\} +\\
&\quad + 2\partial_{xX}\partial_\tau\left\{-T_3\nabla^2 + T_1 + Q\left[(\sigma_1+\sigma_4)\nabla^2 - \sigma_1e_4 - \sigma_4e_1\right]\partial_{zz}-\right.\\
&\quad \left.-R_{oc}[\psi_4e_2 - \sigma_2\mathcal{D}_\psi + \psi_2\partial_{xx}]\right\} + \partial_{\tau\tau}\left\{T_2 - Q\sigma_1\sigma_4\nabla^2\partial_{zz}\right\}\nabla^2-\\
&\quad - R_{oc}\sigma_2\psi_4\partial_{xx},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F}_1 &= \left\{\left[T_1 - (\sigma_4e_1 + \sigma_1e_4)\,Q\partial_{zz}\right]\nabla^2 - R_{oc}\left[\psi_4e_2 + \sigma_2\mathcal{D}_\psi\right]\partial_{xx}\right\}\\
\mathcal{F}_2 &= E_4 - E_3\nabla^2 + Q[(e_1+e_4)\,\nabla^2 - e_4e_1]\partial_{zz}+\\
&\quad + R_{oc}[\psi_1e_2\partial_{xx} + \mathcal{D}_\psi\,(\partial_{xx} - e_2)],\\
\mathcal{D}_\psi &= \left[\psi\left(\sigma_4e_1 - \nabla^2\right) + e_4\right],
\end{aligned}$$

and

$$\begin{aligned}
E_2 &= [e_3(e_1+e_4) + e_1(e_4+e_2) + e_2(e_4+e_3)],\\
E_3 &= e_1e_3(e_4+e_2) + e_2e_4(e_1+e_3), \quad E_4 = e_1e_2e_3e_4,\\
T_1 &= (\sigma_2e_4 + \sigma_4e_2)e_1e_3 + (\sigma_3e_1 + e_3)e_2e_4,\\
T_2 &= \sigma_4\,(\sigma_2e_1 + \sigma_1e_2)\,e_3 + \sigma_3\,(\sigma_2e_4 + \sigma_4e_2)\,e_1 + \sigma_1\,(\sigma_2e_3 + \sigma_3e_2)\,e_4,\\
T_3 &= [e_1e_2\,(\sigma_3+\sigma_4) + e_1e_3\,(\sigma_2+\sigma_4) + e_1e_4(\sigma_2+\sigma_3)+\\
&\quad + e_2e_3(\sigma_1+\sigma_4) + (\sigma_1+\sigma_3)e_2e_4 + e_3e_4\,(\sigma_1+\sigma_2)].
\end{aligned}$$

Eq. (5.7a) is a linear problem. We get critical Rayleigh number for the onset of oscillatory convection by using the zeroth order solution w_0 in Eq. (5.7a). At $O(\epsilon^2)$, $\mathcal{N}_0 = 0$ and $\mathcal{L}_1 w_0 = 0$ gives

$$\partial_\tau A_{1L} - v_g \partial_X A_{1L} = 0 \quad \text{and} \quad \partial_\tau A_{1R} + v_g \partial_X A_{1R} = 0, \tag{5.8}$$

where $v_g = (\partial\omega/\partial q)_{q=q_{oc}}$ is the group velocity and is real. Hence from Eq. (5.7b), we get $w_1 = 0$. From equation of continuity we find that $u_1 = 0$. Substituting the zeroth order and first order approximations into (4.14) and (4.15) we get

$$\begin{aligned}
\theta_1 &= -\pi \left[\left(|A_{1R}|^2 + |A_{1L}|^2\right) t_3 + J_1 + J_1^* - c.c\right] \sin 2\pi z, \\
v_1 &= 0, \quad H_{y_1} = 0, \\
C_1 &= -\pi \left[\left(|A_{1R}|^2 + |A_{1L}|^2\right) t_3 + J_2 + J_2^* - c.c\right] \sin 2\pi z, \\
H_{x_1} &= \frac{i\pi\sigma_2}{q_{oc}} \left[\frac{|A_{1R}|^2}{2e_2} + \frac{|A_{1L}|^2}{2e_2^*} + J_3 + J_3^* + c.c.\right] \sin 2\pi z, \\
H_{z_1} &= -2\pi^2\sigma_2 \left[\frac{A_{1R}A_{1L}e^{2iq_{oc}x}}{2e_2 q_{oc}^2} + J_4 + J_4^* - c.c\right] \cos 2\pi z.
\end{aligned} \tag{5.9}$$

The Eq. (5.7c) is solvable when $\mathcal{L}_0 w_0 = 0$, one requires that its right hand side be orthogonal to w_0, which is ensured that if the coefficients of $\sin\pi z$ in $\mathcal{N}_1 - \mathcal{L}_2 w_0$ are equal to zero. This implies that

$$\begin{aligned}
&\Lambda_0 \partial_T A_{1L} + \Lambda_1 \left(\partial_\tau - v_g \partial_X\right) A_{2L} - \Lambda_2 \partial_{XX} A_{1L} - \Lambda_3 A_{1L} + \\
&\qquad + \Lambda_4 \left|A_{1L}\right|^2 A_{1L} + \Lambda_5 \left|A_{1R}\right|^2 A_{1L} = 0,
\end{aligned} \tag{5.10a}$$

$$\begin{aligned}
&\Lambda_0 \partial_T A_{1R} + \Lambda_1 \left(\partial_\tau + v_g \partial_X\right) A_{2R} - \Lambda_2 \partial_{XX} A_{1R} - \Lambda_3 A_{1R} + \\
&\qquad + \Lambda_4 \left|A_{1R}\right|^2 A_{1R} + \Lambda_5 \left|A_{1L}\right|^2 A_{1R} = 0,
\end{aligned} \tag{5.10b}$$

where

$$\begin{aligned}
\Lambda_0 &= \delta_{oc}^2 T_1 + Q_1 \delta_{oc}^2 \left[\sigma_4 e_1 + \sigma_1 e_4\right] - R_{oc} q_{oc}^2 \left[\sigma_4 e_2 + \sigma_2 D_\psi\right], \\
\Lambda_1 &= -\delta_{oc}^2 T_2 - Q_1 \delta_{oc}^2 \sigma_1 \sigma_4 + \sigma_2 \sigma_4 R_{oc} q_{oc}^2 (1 + \psi),
\end{aligned}$$

$$
\begin{aligned}
\Lambda_2 &= 4q_{oc}^2\left\{\delta_{oc}^2 E_2 + E_3 + Q_1\left[e_4 + e_1 + \delta_{oc}^2\right] - R_{oc}[\psi_1(q_{oc}^2 + e_2) + D_\psi]\right\} + \\
&\quad + v_g^2\Lambda_1 + 2iq_{oc}v_g\left\{T_1 + \delta_{oc}^2 T_3 + Q_1\left[\delta_{oc}^2(\sigma_1+\sigma_4) + \sigma_1 e_4 + \sigma_4 e_1\right] - \right.\\
&\quad \left. - R_{oc}\left[\psi_4 e_2 + \sigma_2 D_\psi + q_{oc}^2\psi_2\right]\right\} + E_4 + \delta_{oc}^2 E_3 + Q_1\left[\delta_{oc}^2\left(e_1+e_4\right) + \right.\\
&\quad \left. + e_1 e_4\right] - R_{oc}[q_{oc}^2 e_2\psi_1 + \left(q_{oc}^2 + e_2\right) D_\psi],\\
\Lambda_3 &= R_{oc}q_{oc}^2 e_2 D_\psi,\\
\Lambda_4 &= -\frac{R_{oc}q_{oc}^2\delta_{oc}^2 e_2}{2e_1e_1^*}[\sigma_1 e_4 + \psi\pi^2(e_1\sigma_4 + \sigma_1)] + \\
&\quad + \frac{3Q_1\pi^2\delta_{oc}^2\sigma_2^2 e_1 e_4}{2(\sigma_2+\sigma_3)}\left(\frac{1}{2\pi^2} + \frac{1}{e_q}\right),\\
\Lambda_5 &= -R_{oc}q_{oc}^2 e_2\left(e_4 + \psi\delta_{oc}^2\pi^2\right)\left(\frac{2}{e_1 e_d} + t_3\right) + \\
&\quad + R_{oc}q_{oc}^2\pi^2\psi\sigma_4 e_1 e_2\left(\frac{2}{e_1 e_d b_4} + t_3\right) + 3Q_1\pi^2\delta_{oc}^2\sigma_2^2 e_1 e_4 t_4,
\end{aligned}
\tag{5.11}
$$

with

$$
\begin{aligned}
e_q &= \left(2q_{oc}^2 + i\omega_{oc}\sigma_2\right), \quad e_d = \left(\delta_{oc}^2 + 2i\omega_{oc}\right),\\
b_j &= \left(1 + i\omega_{oc}\sigma_j/2\pi^2\right), \quad j = 1,2,4. \quad b_4 = \left(2\pi^2 + i\omega_{oc}\sigma_4\right),\\
J_1 &= A_{1L}A_{1R}^*\frac{2}{e_1 e_d}\exp\left(2i\omega_{oc}t\right), \quad J_2 = A_{1L}A_{1R}^*\frac{2}{e_1 e_d\, b_4}\exp\left(2i\omega_{oc}t\right),\\
J_3 &= A_{1R}A_{1L}^*\frac{1}{2e_2b_2^*}\exp\left(2i\omega_{oc}t\right), \quad J_4 = (A_{1R}^2)\frac{1}{2e_2e_q^*}\exp\left(q_{oc}x - i\omega_{oc}t\right),\\
t_1 &= \frac{1}{4\pi^2}\left[\sigma_4(h_1 + h_1^*)\right] + t_3, \quad t_2 = \left[2\sigma_4 h_1 + \frac{1}{e_1^* b_1^*}\right],\\
t_3 &= \frac{1}{4\pi^2}\left(\frac{1}{e_1} + \frac{1}{e_1^*}\right),\\
t_4 &= \frac{1}{4}\left[\frac{1}{(\sigma_1+\sigma_4)}(\frac{1}{q_{oc}^2} + \frac{1}{\pi^2}) + \frac{1}{(\sigma_2+\sigma_3)}(\frac{1}{q_{oc}^2} + \frac{1}{b_2\pi^2})\right],\\
D_\psi &= \left[\psi\left(\sigma_4 e_1 + \delta_{oc}^2\right) + e_4\right].
\end{aligned}
$$

All starred quantities e_1^*, e_2^*, e_3^*, e_4^*, t_1^*, t_2^*, $J_1^*, J_2^*, J_3^*, J_4^*$ are complex conjugates of $e_1, e_2, e_3, e_4, t_1, t_2, J_1, J_2, J_3, J_4$. All above mentioned quantities $f = (e_1, e_2, \dots)$ are functions of δ_{oc} and ω_{oc}, i.e. $f = f(\delta_{oc}, \omega_{oc})$. It should be noted that A_{1L}, A_{1R} are of order ϵ and A_{2L}, A_{2R} are of order ϵ^2. If $\omega_{oc} = 0$ in $\Lambda_0, \Lambda_2, \Lambda_3$ and Λ_4 then these expressions match with the coefficients $\lambda_0, \lambda_1, \lambda_2,$ and λ_3 of Landau-Ginzburg equation at the onset of stationary convection.

From Eqs. (5.8), we get $A_{1L}(\xi', T)$ and $A_{1R}(\eta', T)$, where $\xi' = v_g\tau + X$, $\eta' = v_g\tau - X$. Eqs. (5.10a), (5.10b) can be written as

$$2v_g\Lambda_1\partial_{\eta'}A_{2L} = -\Lambda_0\partial_T A_{1L} + \Lambda_2\partial_{XX}A_{1L} + \Lambda_3 A_{1L} - \\ - \left(\Lambda_4 \left|A_{1L}\right|^2 + \Lambda_5 \left|A_{1R}\right|^2\right) A_{1L}, \tag{5.12a}$$

$$2v_g\Lambda_1\partial_{\xi'}A_{2R} = -\Lambda_0\partial_T A_{1R} + \Lambda_2\partial_{XX}A_{1R} + \Lambda_3 A_{1R} - \\ - \left(\Lambda_4 \left|A_{1R}\right|^2 + \Lambda_5 \left|A_{1L}\right|^2\right) A_{1R}. \tag{5.12b}$$

Let $\xi' \epsilon [0, l_1]$, $\eta' \epsilon [0, l_2]$, where $l_1,\ l_2$ are periods of $A_{1L},\ A_{1R}$, respectively. Expansion (5.4) remains asymptotic for times $t = O(\epsilon^{-2})$ only if an appropriate solvability condition holds. This condition obtained integrating Eq. (5.12a) over η' and Eq. (5.12b) over ξ', we get

$$\Lambda_0\partial_T A_{1L} = \Lambda_2\partial_{XX}A_{1L} + \Lambda_3 A_{1L} - \left(\Lambda_4 \left|A_{1L}\right|^2 + \Lambda_5 \left|A_{1R}\right|^2\right) A_{1L}, \tag{5.13a}$$

$$\Lambda_0\partial_T A_{1R} = \Lambda_2\partial_{XX}A_{1R} + \Lambda_3 A_{1R} - \left(\Lambda_4 \left|A_{1R}\right|^2 + \Lambda_5 \left|A_{1L}\right|^2\right) A_{1R}. \tag{5.13b}$$

5.1. Travelling wave and standing wave convection

To study the stability regions of travelling waves and standing waves we proceed as follows:

On dropping slow space variable X from Eqs. (5.13a) and (5.13b), we get a pair of first order ODE's

$$\frac{dA_{1L}}{dT} = \frac{\Lambda_3}{\Lambda_0}A_{1L} - \frac{\Lambda_4}{\Lambda_0}A_{1L}\left|A_{1L}\right|^2 - \frac{\Lambda_5}{\Lambda_0}A_{1L}\left|A_{1R}\right|^2, \tag{5.14}$$

$$\frac{dA_{1R}}{dT} = \frac{\Lambda_3}{\Lambda_0}A_{1R} - \frac{\Lambda_4}{\Lambda_0}A_{1R}\left|A_{1R}\right|^2 - \frac{\Lambda_5}{\Lambda_0}A_{1R}\left|A_{1L}\right|^2. \tag{5.15}$$

Put

$$\beta' = \frac{\Lambda_3}{\Lambda_0},\ \gamma' = -\frac{\Lambda_4}{\Lambda_0} \text{ and } \delta' = -\frac{\Lambda_5}{\Lambda_0}.$$

Then Eqs. (5.14) and (5.15) take the following form

$$\frac{dA_{1L}}{dT} = \beta^{'} A_{1L} + \gamma^{'} A_{1L} \left|A_{1L}\right|^2 + \delta^{'} A_{1L} \left|A_{1R}\right|^2, \tag{5.16}$$

$$\frac{dA_{1R}}{dT} = \beta^{'} A_{1R} + \gamma^{'} A_{1R} \left|A_{1R}\right|^2 + \delta^{'} A_{1R} \left|A_{1L}\right|^2. \tag{5.17}$$

Consider $A_{1L} = a_L e^{i\phi_L}$ and $A_{1R} = a_R e^{i\phi_R}$ (we can write a complex number in the amplitude and phase (angle) form), where $a_L = |A_{1L}|$, $\phi_L = \arg(A_{1L}) = \tan^{-1}[\mathrm{Im}\, m(A_{1L})/\mathrm{Re}\, e(A_{1L})]$ and $a_R = |A_{1R}|$, $\phi_R = \arg(A_{1R}) = \tan^{-1}(\mathrm{Im}\, m(A_{1R})/\mathrm{Re}\, e(A_{1R}))$. a_L, a_R, ϕ_L, ϕ_R are functions of time T since A_{1L} and A_{1R} are functions of T. Thus a_L and a_R are positive functions. Substituting the definitions of A_{1L} and A_{1R} and $\beta^{'} = \beta_1 + i\beta_2$, $\gamma^{'} = \gamma_1 + i\gamma_2$, $\delta^{'} = \delta_1 + i\delta_2$ into Eqs. (5.16) and (5.17), we get

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L \left|a_L\right|^2 + \delta_1 a_L \left|a_R\right|^2, \tag{5.18}$$

$$\frac{d\phi_L}{dT} = \beta_2 + \gamma_2 \left|a_L\right|^2 + \delta_2 \left|a_R\right|^2, \tag{5.19}$$

$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R \left|a_R\right|^2 + \delta_1 a_R \left|a_L\right|^2, \tag{5.20}$$

$$\frac{d\phi_R}{dT} = \beta_2 + \gamma_2 \left|a_R\right|^2 + \delta_2 \left|a_L\right|^2. \tag{5.21}$$

Eqs. (5.18) and (5.20) not contain phase term, so we take these two equations for the future discussions. We have Eqs. (5.18) and (5.20) as

$$\frac{da_L}{dT} = \beta_1 a_L + \gamma_1 a_L^3 + \delta_1 a_L a_R^2,$$

$$\frac{da_R}{dT} = \beta_1 a_R + \gamma_1 a_R^3 + \delta_1 a_R a_L^2,$$

since a_L and a_R are positive functions. Put

$$\frac{da_L}{dT} = F_1(a_L, a_R), \quad \frac{da_R}{dT} = F_2(a_L, a_R) \tag{5.22}$$

Now we discuss the stability of equilibrium points of above Eqs. (5.22). We get four equilibrium points like $(a_L, a_R) = (0, 0)$ [conduction state], $(a_L, a_R) = (a_L, 0)$ [a_L = amplitude of left travelling waves, here we get $F_2 =$

0, and we get one condition from $F_1 = 0$, i.e., $a_L^2 = -\beta_1/\gamma_1 \left(= |A_{1L}|^2\right)$], $(a_L, a_R) = (0, a_R)$ [a_R = amplitude of right travelling waves, here $F_1 = 0$ and from $F_2 = 0$, we get $a_R^2 = -\beta_1/\gamma_1 \left(= |A_{1R}|^2\right)$], and for $a_L \neq 0$ and $a_R \neq 0$ we get $(a_L, a_R) = (-\beta_1/(\gamma_1 + \delta_1), -\beta_1/(\gamma_1 + \delta_1))$ [this gives condition for standing waves. At standing waves we have $A_{1L} = A_{1R}$, so $a_L = a_R$]. For the pair of Eqs. (5.14) and (5.15), we do not get $a_L \neq a_R \neq 0$ [modulated waves]. Now the Jacobian of F_1 and F_2 is given by

$$\begin{pmatrix} \partial F_1/\partial a_L & \partial F_1/\partial a_R \\ \partial F_2/\partial a_L & \partial F_2/\partial a_R \end{pmatrix}.$$

If real parts of all eigenvalues of the Jacobian are negative at an equilibrium point, then that point is a stable equilibrium [Lyapounov's theorem or principle of linearized stability]. Some valuable conditions for travelling waves and standing waves are: Travelling waves are stable if $\beta_1 > 0,\ \gamma_1 < 0$ and $\delta_1 < \gamma_1 < 0$. Standing waves are stable if $\beta_1 > 0,\ \gamma_1 < 0$ and (i) if $\delta_1 > 0$, then $-\gamma_1 > \delta_1 > 0$, (ii) if $\delta_1 < 0$, then $-\gamma_1 > -\delta_1 > 0$.

The stability regions of travelling waves and standing waves are summarized in Fig. 6. Here E is total amplitude and defined as $E = a_L^2 + a_R^2$. We do not distinguish between left travelling waves and right travelling waves. For rest state (steady state) $E = 0$, for travelling waves $E = -\beta_1/\gamma_1$, for standing waves $E = -2\beta_1/(\gamma_1 + \varsigma_1)$. Travelling waves are supercritical if $\gamma_1 < 0$ and standing waves are supercritical if $\gamma_1 + \varsigma_1 < 0$. Fig. 6(a) is drawn for stable travelling wave conditions and Fig. 6(b) is drawn for stable standing wave conditions in (β_1, E)-plane. The symbols $(-,-)$ and $(+,-)$ in Figs. 6(a,b) indicate that both two roots of Jacobian are negative and at least one root is positive between two roots.

In Figs. 6(a) and 6(b), travelling wave solution and standing wave solution bifurcate simultaneously from the steady state solution ($\beta_1 \geq 0$ at this bifurcation point). In these Figs. 6(a) and 6(b), steady state solution is stable for $\beta_1 < 0$ and unstable for $\beta_1 > 0$. These figures show that for $\beta_1 > 0$ both travelling waves and standing waves are supercritical. When travelling waves and standing waves bifurcate supercritically then at most one solution among travelling waves and standing waves will be stable. Thus, for

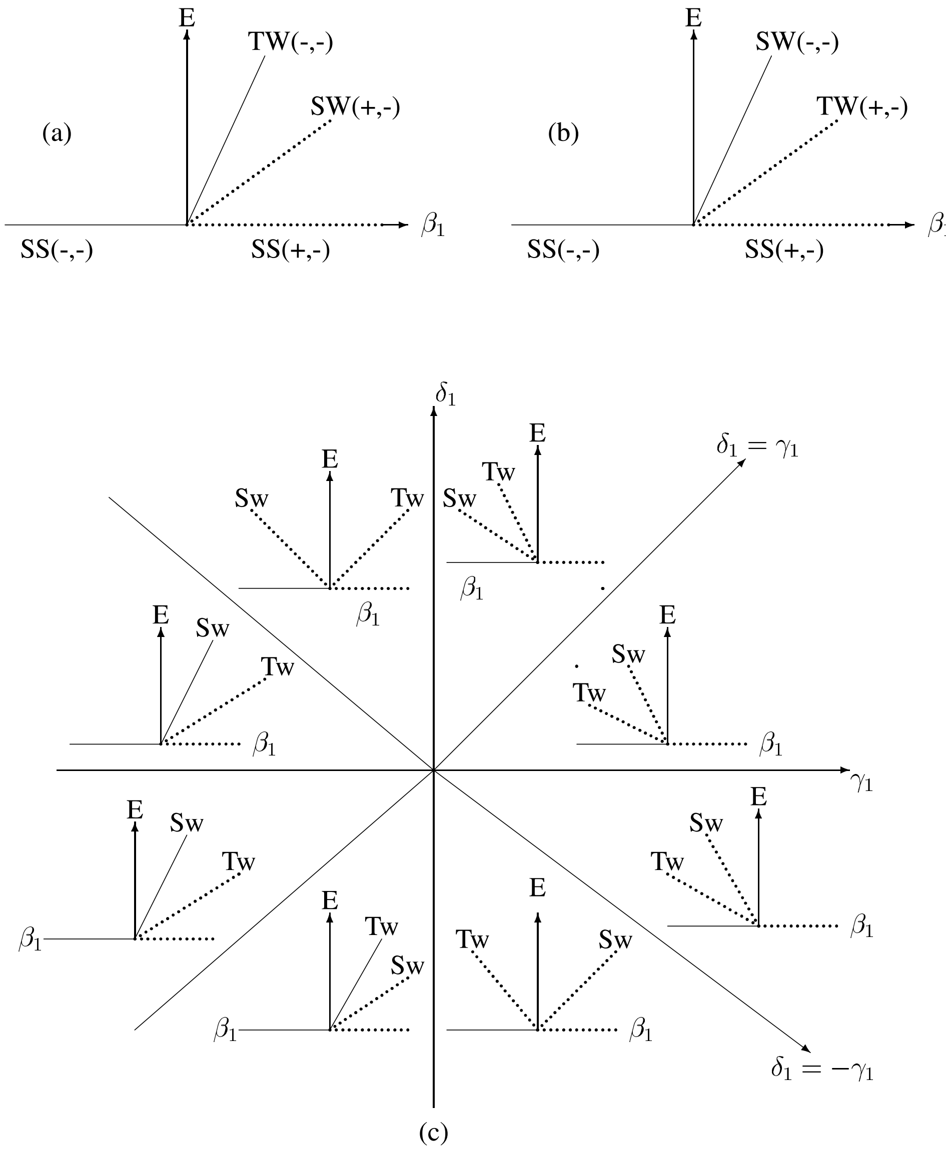

Fig. 6. (a), (b) and (c) are typical diagrams showing the stability of equilibrium solutions SS (steady state), SW(standing waves) and TW(travelling waves). The equilibrium solutions are stable on solid lines and unstable on dotted lines.

$\beta_1 > 0$ (Fig. 6(a)) travelling waves are stable and (Fig. 6(b)) standing waves are stable. In more detail we reproduce results of the stability analysis of equilibrium solutions in Fig. 6(c), which is plotted in (γ_1, ς_1)-plane. From this figure we can observe that travelling waves are subcritical for $\gamma_1 > 0$ and standing waves are subcritical for $\gamma_1 + \varsigma_1 > 0$.

5.2. Long wave-length instabilities for the onset of travelling wave convection (Benjamin-Feir instability)

For right travelling wave $A_R(X,T) = A(X,T)$ and $A_L(X,T) = 0$, for left travelling wave $A_R(X,T) = 0$ and $A_L(X,T) = A(X,T)$. Thus for travelling waves we get a single amplitude equation from Eqs. ((5.10a), (5.10b), given as

$$\Lambda_0 \partial_T A - \Lambda_2 \partial_{XX} A - \Lambda_3 A + \Lambda_4 |A|^2 A = 0, \tag{5.23}$$

For standing waves $A_{1L}(X,T) = A_{1R}(X,T) = A(X,T)$ and we get a single amplitude equation from Eqs. (5.10a) and (5.10b), given as

$$\Lambda_0 \partial_T A - \Lambda_2 \partial_{XX} A - \Lambda_3 A + (\Lambda_4 + \Lambda_5) |A|^2 A = 0. \tag{5.24}$$

The above Eq. (5.23) possesses a family of planar wave solutions and solutions containing phase singular points.

We study the Benjamin-Feir instability of travelling waves from complex Landau-Ginzburg Eq. (5.23). Eq. (5.23) can be written as

$$\partial_T A = \xi \partial_{XX} A + \beta A + \gamma |A|^2 A, \tag{5.25}$$

where $\xi = \xi_1 + i\xi_2,\ \beta = \beta_1 + i\beta_2,\ \gamma = \gamma_1 + i\gamma_2$. The phase winding solutions are obtained by substituting

$$A = \tilde{A}_o e^{i(\delta q_o X - \delta\omega T)}$$

into Eq. (5.25) and equating real and imaginary parts we get

$$|\tilde{A}_o|^2 = \xi_1 \delta q_o^2 - \beta_1 \gamma_1^{-1}, \delta\omega = \xi_2 \delta q_o^2 - \beta_2 + \gamma_2 \left(\beta_1 - \xi_1 \delta q_o^2\right) \gamma_1^{-1}.$$

Here $\tilde{A}_o$ is a constant and $\delta q_o = q_X - q_{oc}$. We consider a modulated solution in the form: $A(X,T) = \tilde{A}(X,T) e^{i(\delta q_o X - \delta\omega T)}$. Substituting the modulated solution into Eq. (5.25) which gives

$$\partial_T \tilde{A} = (\gamma_1 + i\gamma_2)\left[\left(\beta_1 - \delta q_o^2 \xi_1 \gamma_1^{-1}\right) + |\tilde{A}|^2\right]\tilde{A} + \\ + (\xi_1 + i\xi_2)\left(\partial_{XX} + 2i\delta q_o \partial_X\right)\tilde{A}. \tag{5.26}$$

It is possible to conduct a general investigation of the linear stability of $A(X,T)$, but this is a very difficult task, and therefore our primary concern here is to treat the stability of the uniformly oscillating solution $\tilde{A}_o$. Inserting $\tilde{A} = \tilde{A}_o + \tilde{u} + i\tilde{v}$ into Eq. (5.26) and equating real and imaginary parts we get

$$\partial_T \tilde{u} = -2\left(\beta_1 - \delta d q_o^2 \xi_1\right)\tilde{u} + \xi_1\left(\partial_{XX}\tilde{u} - 2\delta q_o \partial_X \tilde{v}\right) - \\ - \xi_2\left(2\delta q_o \partial_X \tilde{u} + \partial_{XX}\tilde{v}\right), \tag{5.27a}$$

$$\partial_T \tilde{v} = -2\gamma_2\left(\beta_1 - \delta q_o^2 \xi_1\right)\gamma_1^{-1}\tilde{u} + \xi_1\left(2\delta q_o \partial_X \tilde{u} + \partial_{XX}\tilde{v}\right) + \\ + \xi_2\left(\partial_{XX}\tilde{u} - 2\delta q_o \partial_X \tilde{v}\right). \tag{5.27b}$$

Consider $(\tilde{u}, \tilde{v}) = (U, V)e^{ST}\cos q_X X$ and S is the growth rate of disturbances. Using solutions of $\tilde{u}$, $\tilde{v}$ and $\delta q_o = 0$ into (5.32), we get

$$\left(S + 2\beta_1 + \xi_1 q_X^2\right)U - q_X^2 \xi_2 V = 0, \tag{5.28}$$

$$\left(S + q_X^2 \xi_1\right)V + \left(2\beta_1 \gamma_2 \gamma_1^{-1} + q_X^2 \xi_2\right)U = 0. \tag{5.29}$$

Solving Eq. (5.28) and Eq. (5.29), we get

$$S^2 + 2S\left(\beta_1 + \xi_1 q_X^2\right) + q_X^2 \xi_1\left(2\beta_1 + \xi_1 q_X^2\right) + q_X^2 \xi_2\left(2\beta_1 \gamma_2 \gamma_1^{-1} + q_X^2 \xi_2\right) = 0. \tag{5.30}$$

There will be an instability only when a root of Eq. (5.30) is positive i.e.,

$$2\beta_1\left(\xi_1 + \gamma_2 \xi_2 \gamma_1^{-1}\right) + q_X^2\left(\xi_1^2 + \xi_2^2\right) < 0. \tag{5.31}$$

$\beta_1 > 0$ when travelling waves or standing waves are stable. The instability of waves against long wavelength longitudinal modes is often called the Benjamin-Feir instability. Thus we get Benjamin-Feir instability for travelling waves when $\xi_1 + \gamma_2\xi_2/\gamma_1 < 0$. Similarly by considering Eq. (5.25) instead of Eq. (5.24) and proceeding in the same way we get Benjamin-Feir instability for standing waves when $\xi_1 + (\gamma_2 + \delta_2)\xi_2/(\gamma_1 + \delta_1) < 0$.

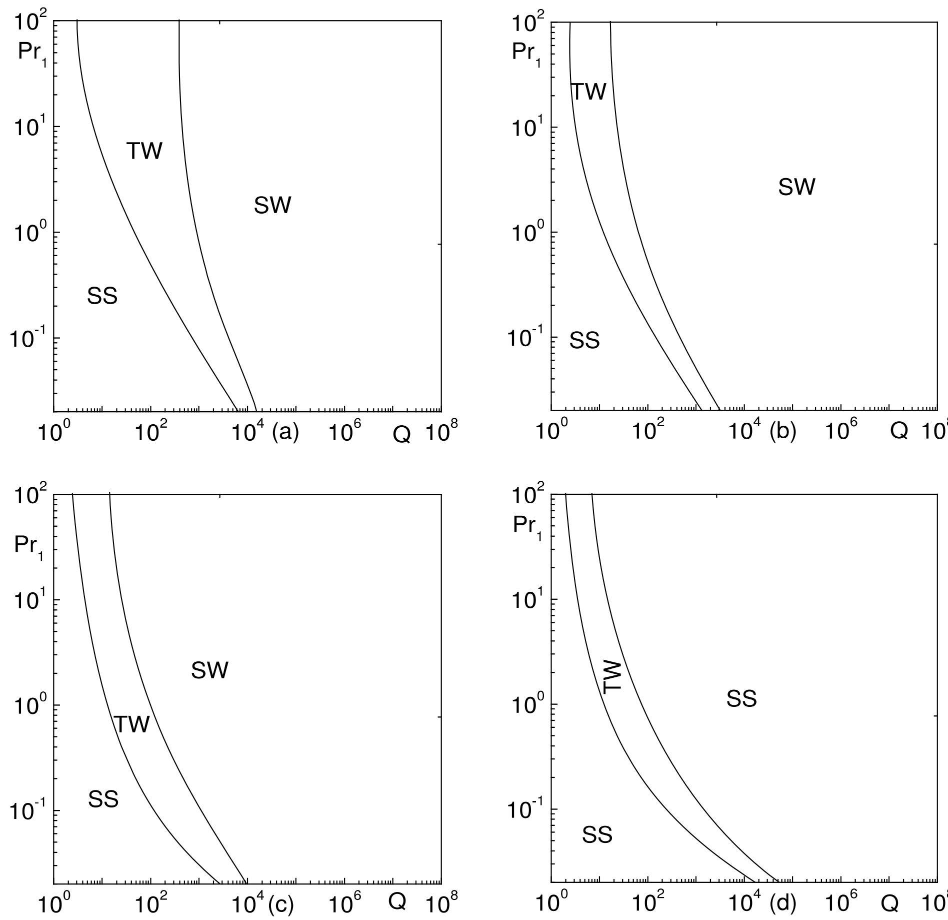

Fig. 7. Figures (a–d), are plotted for $\sigma_2 = 6, 12, 18, 24$, respectively. Stability regions of steady state (SS), travelling waves (TW) and standing waves (SW) are plotted (Q, Pr_1)-plane.

6. Conclusions

In this paper we have considered both the linear and the weakly nonlinear analysis of magneto-convection in Earth's outer core due to compositional and thermal buoyancy by using free-free (stress-free) boundary conditions.

Even though the free-free boundary conditions cannot be achieved in laboratory, one can use it in geophysical fluid dynamic applications to Earth's outer core, since they allow simple trigonometric eigenfunctions.

Following *Chandrasekhar (1961)*, we have described the stationary convection and oscillatory convection as curves $R_s(\psi)$ and $R_o(\psi, \sigma_2)$ vs wave numbers. The critical wave numbers for stationary convection and oscillatory convection are $q_{sc} = q_{oc} = \pi/\sqrt{2}$. For the problem of convection due to compositional and thermal buoyancy, we get Takens-Bogdanov bifurcation point, but no codimension-two bifurcation point. In the non-linear Eq. (4.17), $\lambda_0 = 0$ gives the Takens-Bogdanov bifurcation point at $q_s = q_{sc}$, and when $\lambda_0 = 0$, Eq. (4.17) is not valid. The pitchfork bifurcation is supercritical if $\lambda_3 > 0$, subcritical if $\lambda_3 < 0$, and we get tricritical point if $\lambda_3 = 0$.

From Eq. (4.17), we have obtained in section 4.1 secondary instabilities cf. Eckhaus and zigzag instabilities. We have computed stability regions (see Fig. 7) of SW and TW at both the Hopf bifurcation and the co-dimension two bifurcation point. The conditions for SW and TW are $A_L = A_R$ and $A_L = 0$ or $A_R = 0$, respectively. The TW exist if $|A_L|^2 = -\beta_1/\gamma_1 > 0$, and they are supercritical if $\gamma_1 < 0$. The SW exist if $|A_L|^2 = |A_R|^2 = -\beta_1/(\gamma_1 + \delta_1) > 0$ and SW are supercritical if $\gamma_1 + \delta_1 < 0$. When both the SW and TW are supercritical, then at most one equilibrium solution is stable. If we substitute $\omega = 0$ in the coefficients of Eqs. (5.13a–5.13b), we get the coefficients of Eq. (4.17). At the codimension-two bifurcation point we always get TW. These TW are replaced by SW when Pr increases. At Takens-Bogdanov bifurcation point we get both the TW and SW. By deriving one-dimensional Landau-Ginzburg equations with complex coefficients cf. Eqs. (5.23) and (5.24), we have shown the existence of the Benjamin-Feir type of instability for both TW and SW.

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