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A mathematical model of the bathymetry-generated external gravitational field

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Abstract: The currently available global geopotential models and the global elevation and bathymetry data allow modelling the topography-corrected and bathymetry stripped reference gravity field to a very high spectral resolution (up to degree 2160 of spherical harmonics) using methods for a spherical harmonic analysis and synthesis of the gravity field. When modelling the topography-corrected and crust-density-contrast stripped reference gravity field, additional stripping corrections are applied due to the ice, sediment and other major known density contrasts within the Earth's crust. The currently available data of global crustal density structures have, however, a very low resolution and accuracy. The compilation of the global crust density contrast stripped gravity field is thus limited to a low spectral resolution, typically up to degree 180 of spherical harmonics. In this study we derive the expressions used in forward modelling of the bathymetry-generated gravitational field quantities and the corresponding bathymetric stripping corrections to gravity field quantities by means of the spherical bathymetric (ocean bottom depth) functions. The expressions for the potential and its radial derivative are formulated for the adopted constant (average) ocean saltwater density contrast and for the spherical approximation of the geoid surface. These newly derived expressions are utilized in numerical examples to compute the gravitational potential and attraction generated by the ocean density contrast. The computation is realized globally on a 1×1 arc-deg geographical grid at the Earth's surface.

Key words: bathymetry, Earth gravity field, forward modelling, spherical harmonics

1. Introduction

Various methods have been applied to evaluate the topographic corrections to gravity field quantities. Studying the gravitational contribution of the far-zone topography and the long-wavelength topography corrected reference gravity field, the spectral representation of Newton's integral is commonly utilized in deriving the expressions for the forward modelling of the topography-generated gravitational field. *Sünkel (1968)* derived the expressions for computing the topographic and topographic-isostatic potentials in terms of the spherical height functions. Alternative expressions for the topographic potential and its vertical gradient were formulated in *Vaníček et al. (1995). Sjöberg and Nahavandchi (1999), Tsoulis (1999), Sjöberg (2000), Novák (2000), Novák et al. (2001), Tsoulis (2001), Sjöberg (2001), Heck (2003), Tenzer (2005), Sjöberg (2007), Novák (2009)* and others derived various types of expressions for computing the topography-generated gravitational field quantities using spherical harmonic analysis of the gravitational field. *Wild and Heck (2004)* derived expressions for the topographic effect on satellite gradiometry observations. *Makhloof (2007)* derived expressions for computing the topographic-isostatic effects on airborne gravimetry, satellite gravimetry and gradiometry data. Alternative expressions for computing the topographic effects in satellite gravimetry and gradiometry were formulated by *Novák and Grafarend (2006)* and *Eshagh and Sjöberg (2008, 2009). Novák and Grafarend (2005)* derived the topographic potential and its vertical gradient using the ellipsoidal representation of Newton's integral.

Sjöberg (1993, 1998) and *Sjöberg and Nahavandchi (1999)* defined the atmospheric effects on gravity and geoid using spherical harmonic analysis of the gravitational field. This concept was further developed in *Sjöberg (1999, 2001)* and *Sjöberg and Nahavandchi (2000).* In these studies the geometry of the lower atmospheric bound is described by the coefficients of a global elevation model. *Ramillien (2002)* applied a similar concept to compute the atmosphere-generated gravitational attraction. *Nahavandchi (2004)* used a novel approach to compute the direct atmospheric gravity effect on a regular grid at the Earth's surface over the territory of Iran including offshore areas. He combined the local and global topographic information using the detailed digital terrain models and the global elevation model coefficients. *Sjöberg (2006)* derived the expressions in spectral representation for the atmospheric potential and attraction considering the ellipsoidal layering of the Earth's atmosphere. The computation of atmospheric gravity corrections in satellite geodesy applications was discussed for instance by *Novák and*

Grafarend (2006) and *Eshagh and Sjöberg (2009)*. *Novák and Grafarend (2006)* proposed a method for computing the gravitational effect of atmospheric masses on spaceborne data based on spherical harmonic approach with a numerical study in North America. *Eshagh and Sjöberg (2009)* applied an alternative spherical approach to compute the atmospheric effect on satellite gravity gradiometry data over Fennoscandia. *Tenzer el al. (2009b)* applied the analytical continuation approach in deriving the expressions for modelling the atmospheric corrections to gravity field quantities in the form of the spherical height functions.

In geophysical studies investigating the lithosphere structure (cf. e.g. *Kaban et al., 1999, 2003, 2004; Kaban and Schwintzer 2001*) the gravitational effect of the known (in terms of a model produced as a result of other geoscientific investigations) subsurface mass density distribution is removed from the observed gravity field quantities in order to unmask the remaining gravitational signal of the unknown (and sought) anomalous subsurface density distribution or the density interface. The gravitational field generated by the ocean density contrast (i.e., the bathymetry-generated gravitational field) represents significant amount of the signal to be modelled and subsequently removed from the observed gravity field quantities. *Tenzer et al. (2008a, 2008b, 2009a)* computed globally the bathymetric stripping corrections to gravity field quantities using methods for a spherical harmonic analysis and synthesis of the gravitational field. *Vajda et al. (2008)* investigated the global ellipsoid-referenced bathymetric stripping correction to gravity disturbance. Novák (2009) computed globally the gravitational potential generated by the ocean saltwater density with a high-degree spectral resolution.

In this study, we apply the analytical continuation approach to derive the expressions for modelling the bathymetry-generated gravitational field quantities. The principle of this procedure is based on applying the surface spherical harmonic series to describe the gravitational field quantities for a point at the geoid surface. When the corresponding gravitational field quantities are computed above the geoid, this series is analytically upward continued using a Taylor series. The alternative expressions by means of using the external type series of the solid spherical harmonics were derived by *Novák (2009)*. The different numerical aspects of using the analytical continuation approach and the solid spherical harmonic approach are not discussed in this study. The expressions for computing the bathymetry-generated gravitational potential and attraction by means of the surface spherical bathymetric functions are formulated in Section 2. The numerical examples are shown in Section 3. The summary and conclusions are given in Section 4.

2. The bathymetry-generated gravitational field

To model the bathymetry-generated gravitational field, we consider the spherical approximation of the geoid surface and adopt a constant ocean saltwater density contrast. The ocean saltwater density contrast $\Delta\rho^{\mathrm{w}}$ is defined as the difference of the average crustal density ρ^{crust} and the mean ocean saltwater density ρ^{w}. Let us further define the bathymetry-generated gravitational potential V^b in the point (r, Ω) above the geoid surface by means of the analytical upward continuation of the respective value evaluated at the geoid surface. Hence

$$V^b\,(r,\Omega) = V^b\,(\mathrm{R},\Omega) + \sum_{k=1}^{\infty} \frac{(r-\mathrm{R})^k}{k\,!}\,\frac{\partial^{\mathrm{k}} V^b\,(r,\Omega)}{\partial\, r^{\mathrm{k}}}\bigg|_{r=\mathrm{R}}. \tag{1}$$

Similarly, the bathymetry-generated gravitational attraction g^b in the point (r, Ω) above the geoid surface is written as

$$g^b\,(r,\Omega) = g^b\,(\mathrm{R},\Omega) - \sum_{k=1}^{\infty} \frac{(r-\mathrm{R})^k}{k\,!}\,\frac{\partial^{\mathrm{k+1}} V^b\,(r,\Omega)}{\partial\, r^{\mathrm{k+1}}}\bigg|_{r=\mathrm{R}}. \tag{2}$$

The 3-D position is defined by the geocentric radius r and the geocentric direction $\Omega = (\phi,\ \lambda)$, where ϕ and λ are the geocentric spherical latitude and longitude, respectively. The geocentric radius of the geoid surface is approximated by the Earth's mean radius R. The gravitational potential $V^b\,(\mathrm{R},\Omega)$ and the gravitational attraction $g^b\,(\mathrm{R},\Omega)$ in Eqs. (1) and (2) are evaluated in the point (R,Ω) at the geoid surface. The expressions for computing $V^b\,(\mathrm{R},\Omega)$ and $g^b\,(\mathrm{R},\Omega)$ are derived in spectral form in Eqs. (14) and (16). The radial derivatives of the potential $\left\{ \partial^{\mathrm{k}}\, V^b/\partial\, r^{\mathrm{k}} :\ k = 1,\ 2, ... \right\}$ on the right-hand side of Eqs. (1) and (2) are defined as follows

$$\left.\frac{\partial^{\mathrm{k}} V^b(r,\Omega)}{\partial\, r^{\mathrm{k}}}\right|_{r=\mathrm{R}} =$$

$$= \mathrm{G}\,\Delta\rho^{\mathrm{w}} \iint_{\Omega'\in\Omega_{\mathrm{O}}} \int_{\mathrm{R}-D(\Omega')}^{\mathrm{R}} \left.\frac{\partial^{\mathrm{k}} \ell^{-1}(r,\psi,\, r')}{\partial\, r^{\mathrm{k}}}\right|_{r=\mathrm{R}} r'^2\, \mathrm{d}r' \mathrm{d}\Omega', \tag{3}$$

where G is Newton's gravitational constant, D is the ocean bottom depth, ℓ is the Euclidean spatial distance of two points (r,Ω) and (r',Ω'), ψ is the spherical distance, $d\Omega' = \cos\phi'\, d\phi'\, d\lambda'$ is the infinitesimal surface element on the unit sphere, and Ω_{O} is the full solid angle.

The spectral representation of the reciprocal spatial distance ℓ^{-1} for the external convergence domain $r \geq r'$ reads (e.g., *Hobson, 1931*)

$$\ell^{-1}(r,\psi,\, r') = \frac{1}{r}\sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\psi), \tag{4}$$

where $\mathrm{P_n}$ are the Legendre polynomials of degree n for the argument of cosine of the spherical distance ψ.

From Eq. (4), the radial derivatives of ℓ^{-1} are found to be (cf. *Tenzer, 2005*)

$$\frac{\partial^{\mathrm{k}} \ell^{-1}(r,\psi,\, r')}{\partial\, r^{\mathrm{k}}} = (-1)^k \frac{k!}{r'^{k+1}} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^{n+1+k} \binom{n+k}{k} \mathrm{P_n}(\cos\ \psi)\,. \tag{5}$$

The substitution from Eq. (5) to Eq. (3) yields

$$\left.\frac{\partial^{\mathrm{k}} V^b(r,\Omega)}{\partial\, r^{\mathrm{k}}}\right|_{r=\mathrm{R}} = -\mathrm{G}\,\Delta\rho^{\mathrm{w}} \frac{(-1)^k}{\mathrm{R}^{k-1}} \sum_{n=0}^{\infty} \frac{(n+k)\,!}{n!} \times$$

$$\times \iint_{\Omega'\in\Omega_{\mathrm{O}}} \mathrm{P_n}(\cos\psi) \int_0^{-D(\Omega')} \sum_{n=0}^{\infty} \left(1-\frac{B'}{\mathrm{R}}\right)^{n+2} \mathrm{d}B'\, \mathrm{d}\Omega', \tag{6}$$

where $B' = \mathrm{R} - r'$. Since the expansion of Newton's integral kernel into a series of the Legendre polynomials converges uniformly and absolutely, the interchange of summation and integration in Eq. (6) is permissible (cf. *Moritz, 1980*).

As seen in Eq. (6), the application of the analytical upward continuation in Eqs. (1) and (2) separates the argument of the geocentric radius of the computation point from the volumetric integral. The spherical harmonic analysis is then applied to the radial derivatives of the potential defined in the point (R, Ω) at the geoid surface. The application of the binomial theorem to the term $(1 - B'/\mathrm{R})^{n+2}$ on the right-hand side of Eq. (6) and, subsequently, the integration of Eq. (6) with respect to B' results in

$$\int_0^{-D(\Omega')} \left(1 - \frac{B'}{\mathrm{R}}\right)^{n+2} \mathrm{d}B' \approx -\mathrm{R} \sum_{i=0}^{\infty} \binom{n+2}{i} \left[\frac{D(\Omega')}{\mathrm{R}}\right]^{i+1} \frac{1}{i+1} . \tag{7}$$

Disregarding terms higher than the second degree in Eq. (7), we get

$$\int_0^{-D(\Omega')} \left(1 - \frac{B'}{\mathrm{R}}\right)^{n+2} \mathrm{d}B' \approx$$
$$\approx -D(\Omega') - \frac{D(\Omega')^2}{2\mathrm{R}} (n+2) - \frac{D(\Omega')^3}{6\mathrm{R}^2} (n+2)(n+1) .$$

Equation (6) then becomes

$$\left.\frac{\partial^{\mathrm{k}} V^b(r, \Omega)}{\partial r^{\mathrm{k}}}\right|_{r=\mathrm{R}} \cong \mathrm{G}\Delta\rho^{\mathrm{w}} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^{k-1}} \frac{(n+k)!}{n!} \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} D(\Omega') P_n(\cos\psi) \, \mathrm{d}\Omega' +$$
$$+ \frac{\mathrm{G}\,\Delta\rho^{\mathrm{w}}}{2} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^k} \frac{(n+k)!}{n!} (n+2) \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} D(\Omega')^2 P_n(\cos\psi) \, \mathrm{d}\Omega' +$$
$$+ \frac{\mathrm{G}\,\Delta\rho^{\mathrm{w}}}{6} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^{k+1}} \frac{(n+k)!}{n!} (n+2)(n+1) \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} D(\Omega')^3 P_n(\cos\psi) \, \mathrm{d}\Omega'. \tag{8}$$

We note here that when computing the bathymetry-generated gravitational field quantities with a high accuracy and resolution, particularly in coastal areas, Eq. (8) can readily be reformulated for a higher than the second-degree terms of the binomial series in Eq. (7).

The constituents on the right-hand side of Eq. (8) are further defined in terms of the surface spherical bathymetric functions $\mathrm{D_n}(\Omega)$, $\mathrm{D_n^2}(\Omega)$, and $\mathrm{D_n^3}(\Omega)$, which are defined as follows

$$\mathrm{D_n}(\Omega) = \frac{2n+1}{4\pi} \iint\limits_{\Omega' \in \Omega_{\mathrm{O}}} D(\Omega') \; \mathrm{P_n}(\cos\psi) \; \mathrm{d}\Omega' = \sum_{\mathrm{m=-n}}^{\mathrm{n}} \mathrm{D_{n,m}} \, \mathrm{Y_{n,m}}(\Omega), \tag{9}$$

and

$$\mathrm{D_n^2}(\Omega) = \sum_{\mathrm{m=-n}}^{\mathrm{n}} \mathrm{D^2_{n,m}\, Y_{n,m}}(\Omega), \quad \mathrm{D_n^3}(\Omega) = \sum_{\mathrm{m=-n}}^{\mathrm{n}} \mathrm{D^3_{n,m}\, Y_{n,m}}(\Omega), \tag{10}$$

where $\mathrm{D}_{n,m}$ are numerical coefficients of the Global Bathymetric Model (GBM) of degree n and order m. The surface spherical harmonic functions $\mathrm{Y_{n,m}}(\Omega)$ read (e.g., *Heiskanen and Moritz, 1967*)

$$\mathrm{Y_{n,m}}(\Omega) = \mathrm{P_{n,m}}(\sin\phi) \begin{cases} \cos\ m\,\lambda & (m \geq 0) \\ \sin\ |m|\,\lambda & (m < 0) \end{cases}, \tag{11}$$

where $\mathrm{P_{n,m}}(\sin\phi)$ are the Legendre associated functions of degree n and order m for the argument of sine of the geocentric spherical latitude ϕ.

Combining Eq. (8) with Eqs. (9) and (10) the generic formula for radial derivatives of the potential V^b in terms of the surface spherical bathymetric functions $\mathrm{D_n}(\Omega)$, $\mathrm{D_n^2}(\Omega)$, and $\mathrm{D_n^3}(\Omega)$ is introduced in the following form

$$\begin{aligned}
\left.\frac{\partial^{\mathrm{k}} V^b(r,\Omega)}{\partial r^{\mathrm{k}}}\right|_{r=\mathrm{R}} &\cong 4\pi \mathrm{G}\Delta\rho^{\mathrm{w}} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^{k-1}} \frac{(n+k)!}{n!} \frac{1}{2n+1} \sum_{\mathrm{m=-n}}^{\mathrm{n}} \mathrm{D_{n,m}\, Y_{n,m}}(\Omega) + \\
&+ 2\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^k} \frac{(n+k)!}{n!} \frac{(n+2)}{2n+1} \sum_{\mathrm{m=-n}}^{\mathrm{n}} \mathrm{D^2_{n,m}\, Y_{n,m}}(\Omega) + \\
&+ \frac{2}{3}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^{k+1}} \frac{(n+k)!}{n!} \frac{(n+2)\ (n+1)}{2n+1} \sum_{\mathrm{m=-n}}^{\mathrm{n}} \mathrm{D^3_{n,m}\, Y_{n,m}}(\Omega) \cong \\
&\cong \frac{2}{3}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}} \sum_{n=0}^{\infty} \frac{(-1)^k}{\mathrm{R}^{k+1}} \frac{(n+k)!}{n!} \frac{1}{2n+1} \times \\
&\times \sum_{\mathrm{m=-n}}^{\mathrm{n}} \left[6\mathrm{R^2 D_{n,m}} + 3\mathrm{R}\,(n+2)\,\mathrm{D^2_{n,m}} + (n+2)\ (n+1)\ \mathrm{D^3_{n,m}}\right] \mathrm{Y_{n,m}}(\Omega)\,.
\end{aligned} \tag{12}$$

The substitution from Eq. (12) to Eq. (1) yields the expression for computing the bathymetry-generated gravitational potential V^b in the point (r,Ω) above the geoid surface using the GBM coefficients complete to degree N of the spherical bathymetric functions. It reads

$$V^b(r,\Omega) \cong V^b(\mathrm{R},\Omega) - \frac{2}{3}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}} \frac{r-\mathrm{R}}{\mathrm{R}^2} \sum_{n=0}^{N} \frac{(n+1)}{2n+1} \times$$

$$\times \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \left[6\mathrm{R}^2\,\mathrm{D_{n,m}} + 3\mathrm{R}\,(n+2)\;\mathrm{D^2_{n,m}} + (n+2)\;(n+1)\;\mathrm{D^3_{n,m}} \right] \mathrm{Y_{n,m}}\,(\Omega)\; +$$

$$+\,\frac{1}{3}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}}\,\frac{(r-\mathrm{R})^2}{\mathrm{R}^3}\sum_{n=0}^{N}\frac{(n+1)\;(n+2)}{2n+1}\;\times$$

$$\times \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \left[6\mathrm{R}^2\mathrm{D_{n,m}} + 3\mathrm{R}\,(n+2)\,\mathrm{D^2_{n,m}} + (n+2)\;(n+1)\,\mathrm{D^3_{n,m}} \right] \mathrm{Y_{n,m}}\,(\Omega)\,. \qquad (13)$$

The corresponding value of the potential $V^b\,(\mathrm{R},\Omega)$ is evaluated at the geoid surface using the following equation

$$V^b\,(\mathrm{R},\Omega)\;\cong\;\frac{2}{3\mathrm{R}}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}}\sum_{n=0}^{N}\frac{1}{2n+1}\;\times$$

$$\times \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \left[6\mathrm{R}^2\,\mathrm{D_{n,m}} + 3\mathrm{R}\,(n+2)\;\mathrm{D^2_{n,m}} + (n+2)\;(n+1)\;\mathrm{D^3_{n,m}} \right] \mathrm{Y_{n,m}}\,(\Omega)\,. \quad (14)$$

Inserting from Eq. (12) to Eq. (2), the expression for the bathymetry-generated gravitational attraction g^b evaluated in the point (r,Ω) above the geoid surface is found to be

$$g^b\,(r,\Omega)\;\cong\;g^b\,(\mathrm{R},\Omega) - \frac{2}{3}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}}\,\frac{r-\mathrm{R}}{\mathrm{R}^3}\sum_{n=0}^{N}\frac{(n+1)\;(n+2)}{2n+1}\;\times$$

$$\times \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \left[6\mathrm{R}^2\mathrm{D_{n,m}} + 3\mathrm{R}\,(n+2)\;\mathrm{D^2_{n,m}} + (n+2)\;(n+1)\;\mathrm{D^3_{n,m}} \right] \mathrm{Y_{n,m}}\,(\Omega)\; +$$

$$+\,\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}}\,\frac{(r-\mathrm{R})^2}{3\,\mathrm{R}^4}\sum_{n=0}^{N}\frac{(n+1)\;(n+2)\;(n+3)}{2n+1}\;\times$$

$$\times \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \left[6\mathrm{R}^2\mathrm{D_{n,m}} + 3\mathrm{R}\,(n+2)\,\mathrm{D^2_{n,m}} + (n+2)\;(n+1)\;\mathrm{D^3_{n,m}} \right] \mathrm{Y_{n,m}}\,(\Omega)\,. \qquad (15)$$

The attraction $g^b\,(\mathrm{R},\Omega)$ is evaluated at the geoid surface using the following equation

$$g^b\,(\mathrm{R},\Omega)\;\cong\;\frac{2}{3}\pi\,\mathrm{G}\,\Delta\rho^{\mathrm{w}}\,\frac{1}{\mathrm{R}^2}\sum_{n=0}^{N}\frac{(n+1)}{2n+1}\;\times$$

$$\times \sum_{\mathrm{m}=-\mathrm{n}}^{\mathrm{n}} \left[6\mathrm{R}^2\mathrm{D_{n,m}} + 3\mathrm{R}\,(n+2)\,\mathrm{D^2_{n,m}} + (n+2)\;(n+1)\,\mathrm{D^3_{n,m}} \right] \mathrm{Y_{n,m}}\,(\Omega)\,. \qquad (16)$$

The expressions for the analytical upward continuation of the bathymetry-generated gravitational potential and attraction in Eqs. (13) and (15) comprise the surface spherical bathymetric functions $\mathrm{D_n}(\Omega)$, $\mathrm{D_n^2}(\Omega)$, and $\mathrm{D_n^3}(\Omega)$. Depending on the accuracy requirements higher than third-degree terms can be derived from the generic formula for radial derivatives of the potential V^b in Eq. (12). The investigation of the convergence and optimal truncation of binomial expansions for computing the bathymetry-generated gravitational potential and attraction is out of the scope of this study. These aspects are discussed for instance by *Sun and Sjöberg (2001)*. Equations (13–16) are used in Section 3 to compute the bathymetry-generated gravitational potential and attraction using the GBM coefficients.

3. Numerical examples

The 5×5 arc-min global ocean depths from the ETOPO5 bathymetry and topography database (provided by the NOAA's National Geophysical Data Centre) are used to generate the GBM coefficients. These coefficients are utilized to compute the gravitational potential and attraction generated by the ocean density contrast with a spectral resolution complete to degree 180 of the spherical bathymetric functions. The mean value of the ocean saltwater density contrast 1640 $\mathrm{kg\,m^{-3}}$ (i.e., difference of the reference mean crustal density 2670 $\mathrm{kg\,m^{-3}}$ and the mean ocean saltwater density 1030 $\mathrm{kg\,m^{-3}}$) are adopted. The results are shown in Figs. 1 and 2. The gravitational potential and attraction are computed globally on the equiangular 1 arc-deg geographical grid at the Earth's surface. The gravitational potential due to the ocean density contrast varies from 16390 to 28528 $\mathrm{m^2s^{-2}}$ with the mean of 22376 $\mathrm{m^2s^{-2}}$, and the standard deviation is 3191 $\mathrm{m^2s^{-2}}$. The maxima are located in the central part of the Pacific Ocean and the minima in the western part of the central Eurasia. The gravitational attraction due to the ocean density contrast varies from 129 to 753 mGal with the mean of 327 mGal, and the standard deviation is 161 mGal. The maxima are located above the oceanic trenches (i.e., the ocean-floor depressions with the deepest parts of the ocean) and the minima in the central parts of continental regions. The areas above the oceanic trenches and the convergent ocean to continent tectonic plate boundaries

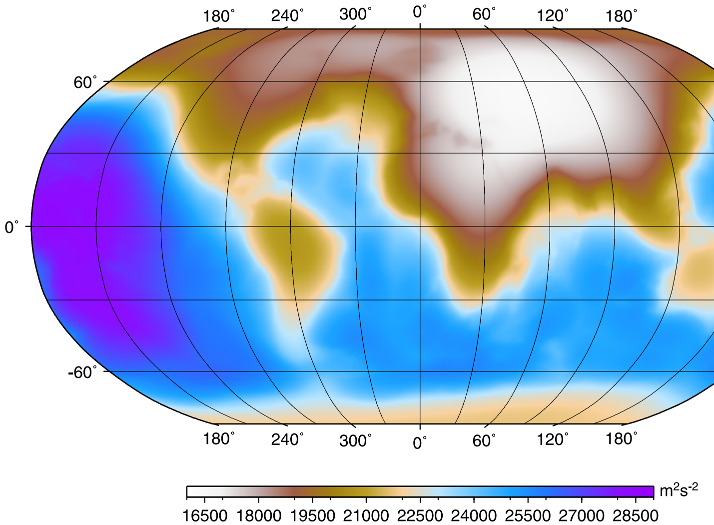

Fig. 1. The gravitational potential due to the ocean density contrast (1640 kg m^{-3}) computed on a 1×1 arc-deg grid at the Earth's surface using the GBM coefficients complete to degree and order 180.

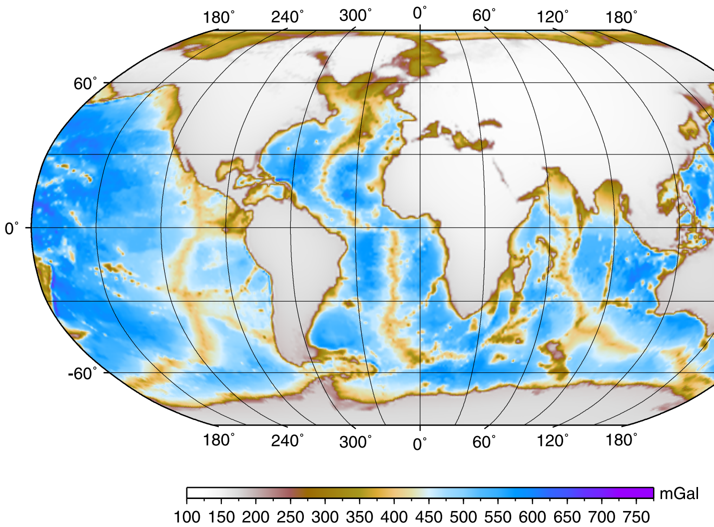

Fig. 2. The gravitational attraction due to the ocean density contrast (1640 kg m^{-3}) computed on a 1×1 arc-deg grid at the Earth's surface using the GBM coefficients complete to degree and order 180.

have the largest variations of the gravitational attraction due to the ocean density contrast.

The actual ocean saltwater density variation due to salinity, temperature and pressure is typically at the interval from 1020 to 1050 $\mathrm{kg\,m^{-3}}$, with most of this range being due to pressure (up to 1.8%). The range of saltwater densities at the sea surface is about 1020 to 1029 $\mathrm{kg\,m^{-3}}$ (cf. *Garrison, 2001*). The anomalous ocean saltwater density variations with respect to the mean value 1030 $\mathrm{kg\,m^{-3}}$ are thus within the interval from –10 to 20 $\mathrm{kg\,m^{-3}}$. Since the errors of the density distribution propagate linearly to the errors of computed gravitational field quantities, the approximation of the actual ocean saltwater density by the mean value may cause large inaccuracies, particularly at the computation points situated over the deep oceans. The errors in computing the gravitational potential due to the ocean density contrast can then reach 550 $\mathrm{m^2s^{-2}}$. The corresponding errors in computing the gravitational attraction can reach 15 mGal. For a more accurate computation, the existing oceanographic models of salinity, temperature and pressure (depth) should be facilitated to determine more realistically the ocean saltwater density distribution. According to *Millero and Poisson (1981)*, the equation of state of seawater nowadays allows calculation of the ocean saltwater density to a fractional accuracy of about 0.03 $\mathrm{kg\,m^{-3}}$.

4. Summary and conclusions

We have derived the expressions for modelling the gravitational field quantities generated by the ocean density contrast in the form of the surface spherical bathymetric functions using methods for a spherical harmonic analysis and synthesis of the gravitational field. The expressions for the gravitational potential and attraction are formulated based on the analytical upward continuation of the respective values evaluated on the geoid surface and utilizing a generic formula for radial derivatives of the potential.

In numerical examples we computed globally the bathymetry-generated gravitational potential and attraction with a low-degree spectral resolution (complete to degree 180 of the spherical bathymetric functions) and adopting the mean value of the ocean saltwater density contrast. The currently available high-resolution bathymetric data combined with the oceanographic salinity and temperature models, however, allow an accurate modelling with a much higher resolution. When the actual ocean saltwater density distribution is approximated by the mean value 1030 $\mathrm{kg\,m^{-3}}$, the relative inaccuracy up to 2% is expected in modelling the bathymetry-generated gravitational field, provided that the actual ocean saltwater anomalous density variations are mainly within –10 to 20 $\mathrm{kg\,m^{-3}}$. The facilitation of existing oceanographic models of salinity, temperature and pressure in modelling the ocean saltwater density distribution is thus indispensable for a more accurate computation of the gravitational field due to the ocean density contrast.

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