Error calibration of quasi-geoidal, normal and ellipsoidal heights of Sweden using variance component estimation

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Abstract: Errors of estimated parameters in an adjustment process should be scaled according to the size of the estimated residuals or misclosures. After computing a quasi-geoid (geoid), its biases and tilts, due to existence of systematic errors in the terrestrial data, are removed by fitting a corrective surface to the misclosures of the differences between the GNSS/levelling data and the quasi-geoid (geoid). Variance component estimation can be used to re-scale or calibrate the error of the GNSS/levelling data and the quasi-geoid (geoid) model. This paper uses this method to calibrate the errors of the recent quasi-geoid model, the GNSS and the normal heights of Sweden. Different stochastic models are investigated in this study and based on a 7-parameter corrective surface model and a three-variance component stochastic model, the calibrated error of the quasi-geoid and the normal heights are 6 mm and 5 mm, respectively and the re-scaled error of the GNSS heights is 18 mm.

Key words: combined adjustment, error calibration, singularity, stochastic model

1. Introduction

Variance component estimation (VCE) is a well-known topic in geodetic and related sciences. Different methods exist for computing the variance components (VCs). One of the most famous methods is the MInimum Norm Quadratic Unbiased Estimator (MINQUE), presented by *Rao (1971)*. Helmert's method is another approach presented by *Kelm (1978)* and by *Grafarend and Schaffrin (1979)*. The method of *LaMotte (1973)* and *Pukelsheim (1981)*, which is very popular in statistics, was generalized by *Schaffrin (1981)* for applications in geodesy. *Horn and Horn (1975)* compared different VC estimators in linear models and *Förstner (1979)* proposed a non-negative VC estimator. *Persson (1980)* studied the MINQUE and related estimator of VCs. *Sjöberg (1983)* investigated an unbiased estimation of VCs in condition adjustment model. *Sjöberg (1985)* presented a VC estimator for the adjustment model with a singular covariance matrix. VCE and applications can also be found in *Rao and Kleffe (1988)*. Maximum likelihood estimation of VCs were first presented for geodetic applications by *Kubik (1970)*, *Patterson and Thompson (1971, 1975)*, and *Koch (1986)*. *Searle et al. (1992)* and *Koch (1999)* provided useful discussions on the concepts of VCs. *Sjöberg (1995)* presented a method for estimating VCs for an additive two – VC model and named the method the best quadratic minimum bias non-negative estimator. As an alternative approach a Monte-Carlo algorithm can be used for VCE (e.g., *Kusche, 2003*). In *Grafarend (2006)* many details of the Gauss-Markov and Gauss-Helmert models are presented, and VCE is treated both theoretically and numerically (see Chapter 4). *Xu et al. (2006)* presented a method for computing VCs in linear ill-posed models. They have considered a zero order Tikhonov regularization method *(Tikhonov, 1963)* for estimating the VCs. They also found out that the simultaneous estimation of the regularization parameter and VCs is advantageous. *Xu et al. (2007)* discussed the estimability of the VCs and proved (as could be expected) that they are not estimable for a fully unknown variance–covariance matrix. *Amiri-Simkooei (2007)* explored a new type of least-squares estimator to the VCs. *Eshagh (2009)* used VCE in direct downward continuation of the satellite gravity gradiometry data for recovering the Earth's gravitational field locally. *Eshagh (2010a)* used VCE for combining the three solutions of gradiometric boundary value problem. Also *Eshagh (2010b)* presented the method of VCE in the discrete ill-posed problems which are solved based on truncated singular value decomposition.

Recently a quasi-geoid model was presented over Sweden by *Ågren et al. (2009)* and named the KTH08 model. This model was computed based on Sjöberg's theory which was developed during last 24 years. This method is known as the least-squares modification of Stokes' formula with additive corrections. More details about the method and its implementations can be found in *Sjöberg (1984a, 1991, 2003)* and *Ellmann (2004, 2005)*, *Ågren (2004)*, *Kiamehr (2006)*. The internal error of the KTH08 quasi-geoid model is 19 mm *(Ågren et al., 2009)*. This error was not calibrated and it was estimated based on propagation of the random errors from the gravitational signal and the error degree variance models used in computing the KTH08. The error of the KTH08 model can be calibrated with the global navigation satellite system (GNSS) data and a VCE process through a combined adjustment of the GNSS heights, normal heights and the KTH08.

Fotopoulos (2003, 2005) was one of those who started using VCE to calibrate the error of geoid models and levelling data. She considered different corrective surfaces to model the discrepancies between geoid models and the geoid computed from the levelling data. Full variance–covariance matrices of geoidal, orthometric and ellipsoidal heights were considered in her analyses. Such matrices can be obtained in different ways; for more details see *Fotopoulos (2003, 2005)*. Another similar work was carried out by *Kiamehr and Eshagh (2008)* for calibrating the error of the gravimetric geoid of Iran. They did not use the full variance–covariance matrices of the heights, but they empirically estimated the error of the geoid, orthometric and ellipsoidal heights. In this paper we shall perform similar error calibration for the KTH08 model. However, this study and its involved problems are different from those performed by others. First of all, we have full covariance matrices of neither the heights nor the quasi-geoid model. Second, the errors of the quasi-geoid model as well as the normal heights are constant for all the points. This yields singularity in the system of equations from which the VCs are estimated. Also the error of the zero-, first- and the second-order GNSS height networks are available. Although the theory of VCE and error calibration of geoid is not new, more practical considerations are needed to do a similar job for the Swedish data. In this respect different stochastic models (SMs) will be defined and used in the VCE process.

2. An overview on combined adjustment and VCE

Consider the following Gauss-Helmert model,

$$\mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\varepsilon} = \mathbf{w}, E\left\{\boldsymbol{\varepsilon}\,\boldsymbol{\varepsilon}^T\right\} = \mathbf{Q} = \sum_{i=1}^{p} \sigma_i^2 \mathbf{Q'}_i \,\text{and}\, E\{\boldsymbol{\varepsilon}\} = 0, \tag{1}$$

where $\mathbf{A}$ and $\mathbf{B}$ are the first and second design matrices of dimensions $(k \times m)$ and $(k \times n)$, $(n \geq k \geq m)$, respectively, $\mathbf{x}$ is the vector of unknowns, $\boldsymbol{\varepsilon}$ is the error vector and $\mathbf{w}$ is the misclosure vector. $\mathbf{Q'}_i$ is the i-th cofactor matrix of the i-th set of observables which subdivides $\mathbf{Q}$ into p parts with one VC for each part, and σ_i^2 is the VC of i-th set of observations. The least-squares solution of Eq. (1) is *(Bjerhammar, 1973)*:

$$\hat{\mathbf{x}} = \left(\mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{A}\right)_{0\mathbf{Q}^{-1}}^{-}\mathbf{w}, \tag{2}$$

and the estimated errors are:

$$\hat{\boldsymbol{\varepsilon}} = \mathbf{B}_{\mathbf{Q}0}^{-}\left[\mathbf{I} - \left(\mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{A}\right)_{0\mathbf{Q}^{-1}}^{0}\right]\mathbf{w}, \tag{3}$$

where $\mathbf{B}_{\mathbf{Q}0}^{-} = \mathbf{Q}\mathbf{B}^{\mathrm{T}}\left(\mathbf{B}\mathbf{Q}\mathbf{B}^{\mathrm{T}}\right)^{-1}$, $\mathbf{B}_{\mathbf{Q}0}^{0} = \mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{B}$ and for any matrix $\mathbf{M}$ we define $\mathbf{M}_{0\mathbf{Q}^{-1}}^{-} = \left(\mathbf{M}^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{M}\right)^{-1}\mathbf{M}^{\mathrm{T}}\mathbf{Q}^{-1}$ and correspondingly $\mathbf{M}_{0\mathbf{Q}^{-1}}^{0} = \mathbf{M}\mathbf{M}_{0\mathbf{Q}^{-1}}^{-}$. For the Gauss-Helmert model the best quadratic unbiased estimate of VCs can be written as (see e.g. *Sjöberg, 1984b*):

$$\hat{\boldsymbol{\sigma}} = \mathbf{S}^{-1}\mathbf{q}, \tag{4a}$$

where $\hat{\boldsymbol{\sigma}}$ is the vector of VCs. The elements of $\mathbf{S}$ and $\mathbf{q}$ are:

$$s_{ij} = \operatorname{trace}\left\{\left[\mathbf{I} - \left(\mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{A}\right)_{0\mathbf{Q}^{-1}}^{0}\right]\left(\mathbf{B}_{\mathbf{Q}0}^{0}\right)^{\mathrm{T}}\mathbf{Q}_i\mathbf{Q}^{-1}\mathbf{Q}_j\mathbf{B}_{\mathbf{Q}0}^{0}\ \times \right.$$
$$\left. \times\left[\mathbf{I} - \left(\mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{A}\right)_{0\mathbf{Q}^{-1}}^{0}\right]\right\}, \tag{4b}$$

and

$$q_i = \mathbf{w}^{\mathrm{T}}\left[\mathbf{I} - \left(\mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{A}\right)_{0\mathbf{Q}^{-1}}^{0}\right]^{\mathrm{T}}\left(\mathbf{B}_{\mathbf{Q}0}^{-}\right)^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{Q}_i\mathbf{B}_{\mathbf{Q}0}^{-}\ \times$$
$$\times\left[\mathbf{I} - \left(\mathbf{B}_{\mathbf{Q}0}^{-}\mathbf{A}\right)_{0\mathbf{Q}^{-1}}^{0}\right]\mathbf{w}. \tag{4c}$$

In calibration of the GNSS/levelling and quasi-geoid errors, the first design matrix $\mathbf{A}$ is selected based on 4-, 5- or 7-parameter model (corrective surface) to remove the biases and tilts of the geoid (quasi-geoid). Equations (5a), (5b) and (5c) are the mathematical models of 4-, 5- and 7-parameter models, respectively (cf. *Fotopoulos, 2003, 2005; Kiamehr, 2006; Kiamehr and Eshagh, 2008*):

$$a_u^{\mathrm{T}} x = x_0 + x_1 \cos\varphi_u \cos\lambda_u + x_2 \cos\varphi_u \sin\lambda_u + x_3 \sin\varphi_u, \tag{5a}$$

$$a_u^{\mathrm{T}} x = x_0 + x_1 \cos\varphi_u \cos\lambda_u + x_2 \cos\varphi_u \sin\lambda_u + x_3 \sin\varphi_u + x_4 \sin^2\varphi_u, \tag{5b}$$

$$\begin{aligned} a_u^{\mathrm{T}} \mathbf{x} = {} & x_0 + x_1 \cos\varphi_u \cos\lambda_u + x_2 \cos\varphi_u \sin\lambda_u + x_3 \sin\varphi_u + \\ & + x_4 \cos\varphi_u \sin\varphi_u \cos\lambda_u / k_u + x_5 \cos\varphi_u \sin\varphi_u \sin\lambda_u / k_u + \\ & + x_6 \sin^2\varphi_u / k_u, \end{aligned} \tag{5c}$$

where φ_u and λ_u are the horizontal geodetic coordinates of the u-th GNSS/levelling point, $k_u = \left(1 - e^2 \sin^2\varphi_u\right)^{1/2}$ and e is the first eccentricity of the reference ellipsoid. x_w, w= 0, 1,..., 7, in Eqs. (5a)–(5c) stand for the transformation parameters which are estimated in the combined adjustment. The main action of these parameters is to remove the biases and tilts between two surfaces of gravimetric and geometric quasi-geoids. The geometric quasi-geoid means the quasi-geoid estimated by subtracting the normal height from its corresponding GNSS height.

As was explained, the structure of the matrix **A**, or in other words, the corrective surface model, is important in the combined adjustment of the gravimetric and the geometric quasi-geoids. One model can present the biases and the tilts better than another. The structure of the second design matrix **B** and type of the SM are not important in the combined adjustment, but in the VCE process they are. In the following section we will discuss the structure of these two matrices.

3. The design matrix B and the SMs

As Eq. (3) shows, **B** connects the error of the observables to the misclosures. It has a direct relation with residuals of each set of observations and involves their variance–covariance matrix. The SM plays an important role in VCE as well. Let us consider the following theorem which is closely related to VCE in the combined adjustment (using the Gauss-Helmert model) of the GNSS, normal heights and the quasi-geoid.

Theorem: *If the matrix* $\mathbf{B}$*, in Eq. (1), and the SM have the following structures:*

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & \mathbf{I} & \mathbf{I} & \cdots & \mathbf{I} \end{bmatrix} \quad \textit{and} \quad \mathbf{Q}_i = \mathbf{Q}_j = \mathbf{I},$$

than the matrix $\mathbf{S}$ *in Eq. (4a) from which the VCs are estimated is singular and the VCs are not uniquely estimable.*

Proof. Since $\mathbf{Q}_i = \mathbf{Q}_j = \mathbf{I}$, the elements of system of equations (4a) become:

$$s_{ij} = \mathrm{trace}\left\{ \left[\mathbf{I} - \left(\mathbf{B}^{-}_{\mathbf{I}0} \mathbf{A} \right)^{0}_{0\mathbf{I}} \right] \left(\mathbf{B}^{0}_{\mathbf{I}0} \right)^{\mathrm{T}} \mathbf{B}^{0}_{\mathbf{I}0} \left[\mathbf{I} - \left(\mathbf{B}^{-}_{\mathbf{I}0} \mathbf{A} \right)^{0}_{0\mathbf{I}} \right] \right\}, \tag{6a}$$

and

$$q_i = \mathbf{w}^{\mathrm{T}} \left[\mathbf{I} - \left(\mathbf{B}^{-}_{\mathbf{I}0} \mathbf{A} \right)^{0}_{0\mathbf{I}} \right]^{\mathrm{T}} \left(\mathbf{B}^{-}_{\mathbf{I}0} \right)^{\mathrm{T}} \mathbf{B}^{-}_{\mathbf{I}0} \left[\mathbf{I} - \left(\mathbf{B}^{-}_{\mathbf{I}0} \mathbf{A} \right)^{0}_{0\mathbf{I}} \right] \mathbf{w}, \tag{6b}$$

which show that the elements s_{ij} do not depend on the indices i and j, so that they have the same expression for any choice of i and j. It means that the matrix $\mathbf{S}$ will have the same elements and it will be singular. This is also the case for q_i as it will be independent of i and j.

Based on this theorem, the VCs will not be uniquely estimable in the combined adjustment of the GNSS and normal heights with the quasi-geoid. The question which can be arisen here is how other researchers e.g. *Fotopoulos (2003), Eshagh and Sjöberg (2008)* and *Kiamehr and Eshagh (2008)* estimated the VCs for the similar problems. To answer, we should mention that the main reason of the singularity of Eq. (4a) is due to selection of an identity matrix for the cofactor matrices. If we can select full cofactor matrices, as *Fotopoulos (2003)* did, or if we can have a diagonal matrix with different diagonal elements, as *Kiamehr and Eshagh (2008)* had, the VCs are estimable. However, such information is seldom available. *Fotopoulos (2003)* selected the EGM96 *(Lemoine et al., 1998)* full variance–covariance matrix to propagate it into the geoidal heights and the full variance–covariance matrices of the orthometric and ellipsoidal heights were estimated in a least-squares adjustment of national height networks. *Kiamehr and Eshagh (2008)* estimated different standard errors for the heights empirically. However, in our study we do not have such detailed information about the errors of the heights in Sweden.

4. The combined adjustment of the quasi-geoid model and the GNSS/levelling heights and different SMs

In the combined adjustment of the quasi-geoid and the GNSS/levelling data, Eq. (1) can be rewritten as:

$$\mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\varepsilon} = \mathbf{w} = \mathbf{h}_{\text{GNSS}} - \mathbf{h}_{\text{Normal}} - \boldsymbol{\zeta}, \tag{7}$$

where $\mathbf{h}_{\text{GNSS}}$, $\mathbf{h}_{\text{Normal}}$ and $\boldsymbol{\zeta}$ are the vectors of GNSS, normal and quasi-geoidal heights. Solution of Eq. (7) was given in Eq. (2). As above explained the matrix $\mathbf{A}$ is defined according to the corrective surface model. The matrix $\mathbf{B}$ and the type of the SM are defined according to the data sets. One important point in VCE is the proper choice of SM. Such a model can be selected in different ways according to some a-priori assumptions. Since VCE is usually used in adjustment problems involving heterogeneous data, one can easily assume a SM to estimate one VC for each type of observation. Therefore heterogeneity can be a criterion for selecting a SM. However, the SM can be selected in such a way that one VC is estimated for more than one type of observation. In the following subsections we will introduce different SMs, such as two-, three-, four- and five-VC SMs.

4.1 Two-VC SM

Let the geometric quasi-geoid be denoted by ζ_1 and the gravimetric one by ζ_2. The idea is to estimate the VCs σ_1^2 and σ_2^2 for ζ_1 and ζ_2, respectively. Also this SM can be selected so that one VC is estimated for h_{GNSS} and one for $\hat{h}_{\text{GNSS}}$ from the quasi-geoid and the normal heights. In the latter case σ_1^2 and σ_2^2 will be the VCs of h_{GNSS} and $\hat{h}_{\text{GNSS}}$, respectively. In both cases the matrix $\mathbf{B}$ and the SM have the following structures:

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} \end{bmatrix}, \tag{8a}$$

and

$$\mathbf{Q} = \sigma_1^2 \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \sigma_2^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 \end{bmatrix}, \tag{8b}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the cofactor matrices of ζ_1 and ζ_2 or h_{GNSS} and $\hat{h}_{\mathrm{GNSS}}$, respectively. The dimensions of $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{0}$ (a matrix with zero elements) are the same as that of $\mathbf{I}$ and equal to the number of the GNSS/levelling points.

4.2. Three-VC SM

Consider the case where one VC is estimated for each set of heights, i.e. one for the GNSS heights, one for the normal heights and one for the quasi-geoid. We name this SM three-VC model. In such a model the matrix $\mathbf{B}$ and the SM are:

$$\mathbf{B} = \begin{bmatrix} \mathbf{I} & -\mathbf{I} & -\mathbf{I} \end{bmatrix}, \tag{9a}$$

and

$$\mathbf{Q} = \sigma_1^2 \begin{bmatrix} \mathbf{Q}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \sigma_2^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \sigma_3^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_3 \end{bmatrix}, \tag{9b}$$

where $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are the cofactor matrices of the GNSS, normal and quasi-geoidal heights, respectively and σ_1^2, σ_2^2 and σ_3^2 their corresponding VCs. The dimensions of $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ and $\mathbf{0}$ are equivalent to that of $\mathbf{I}$.

4.3. Four-VC SM

Here we consider one VC for each zero-, first- and second-order network of the GNSS heights and one for the reconstructed GNSS height, $\hat{h}_{\mathrm{GNSS}}$. In other words, we will estimate four VCs. In order to do that we have to reconstruct the coefficient matrix $\mathbf{B}$ and the SM in the following forms:

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}, \tag{10a}$$

and

$$\mathbf{Q} = \sigma_1^2\mathbf{Q'}_1 + \sigma_2^2\mathbf{Q'}_2 + \sigma_3^2\mathbf{Q'}_3 + \sigma_4^2\mathbf{Q'}_4, \tag{10b}$$

where

$$\mathbf{Q'}_1 = \operatorname{diag}\left(\mathbf{Q}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$$

$$\mathbf{Q'}_2 = \operatorname{diag}\left(\mathbf{0}, \mathbf{Q}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$$

$$\mathbf{Q'}_3 = \operatorname{diag}\left(\mathbf{0}, \mathbf{0}, \mathbf{Q}_3, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$$

$$\mathbf{Q'}_4 = \operatorname{diag}\left(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{Q}_4, \mathbf{Q}_5, \mathbf{Q}_6\right).$$

The dimensions of the identity matrices $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_3$ are consistent with number of the GNSS heights in the zero-, first- and second-order networks. In the levelling network of Sweden we have 25 points in the zero-order network then $\mathbf{I}_1$ is a 25×25 matrix and correspondingly we have dimensions of 181×181 and 1364×1364 for $\mathbf{I}_2$ and $\mathbf{I}_3$. The dimensions of $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are equal to that of $\mathbf{Q}_4$, $\mathbf{Q}_5$ and $\mathbf{Q}_6$, respectively.

4.4. Five-VC SM

Now we add one more VC to the previous SM. In this subsection we consider VCs of $h1_{\mathrm{GNSS}}$, $h2_{\mathrm{GNSS}}$ and $h3_{\mathrm{GNSS}}$, h_{Normal} and ζ. In this case, we have to consider the following forms for $\mathbf{B}$ and the SM:

$$\mathbf{B} = \begin{bmatrix} \mathbf{I}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_1 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_2 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 & \mathbf{0} & \mathbf{0} & -\mathbf{I}_3 \end{bmatrix}, \tag{11a}$$

and

$$\mathbf{Q} = \sigma_1^2\mathbf{Q'}_1 + \sigma_2^2\mathbf{Q'}_2 + \sigma_3^2\mathbf{Q'}_3 + \sigma_4^2\mathbf{Q'}_4 + \sigma_5^2\mathbf{Q'}_5, \tag{11b}$$

where

$$\mathbf{Q'}_1 = \operatorname{diag}\left(\mathbf{Q}_1, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$$

$$\mathbf{Q'}_2 = \operatorname{diag}\left(\mathbf{0}, \mathbf{Q}_2, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$$

$$\mathbf{Q'}_3 = \operatorname{diag}\left(\mathbf{0}, \mathbf{0}, \mathbf{Q}_3, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$$

$\mathbf{Q'}_4 = \text{diag}\left(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{Q}_4, \mathbf{Q}_5, \mathbf{Q}_6, \mathbf{0}, \mathbf{0}, \mathbf{0}\right),$

$\mathbf{Q'}_5 = \text{diag}\left(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{Q}_7, \mathbf{Q}_8, \mathbf{Q}_9\right)$

with the following properties

$\dim(\mathbf{Q_1}) = \dim(\mathbf{Q_4}) = \dim(\mathbf{Q_7}) = \dim(\mathbf{I_1}),$

$\dim(\mathbf{Q_2}) = \dim(\mathbf{Q_5}) = \dim(\mathbf{Q_8}) = \dim(\mathbf{I_2}),$

$\dim(\mathbf{Q_3}) = \dim(\mathbf{Q_6}) = \dim(\mathbf{Q_9}) = \dim(\mathbf{I_3}).$

5. Data description and numerical results

Since the matrix **B** and the SM are structured according to the available data sets we divide this section into two parts. In the first part, the data sets are described and in the second part, the combined adjustment and VCE based on the different SMs are numerically investigated.

5.1. Data description

Since the problem of calibrating the quasi-geoid error of Sweden is our main goal, it is worthwhile to introduce its quasi-geoid model and its height networks. Later we will show that we can introduce different SMs based on these types of data.

5.1.1. The KTH08 quasi-geoid model

The theory of the KTH geoid determination method was implemented by *Ågren (2004)* and *Kiamehr (2006)* and the result of these implementations was summarized as the KTH Geolab software. 495545 gravity observations of the Nordic geodetic commission with a resolution of 2 km by 2 km, the digital elevation model SCANDEM 2004 *(Bilker, 2004)* with a resolution of 100 m by 100 m were used in the computation of the KTH08 quasi-geoid model. The global gravitational model CGM02C *(Tapley et al., 2005)* up to degree 200 and the EGM96 *(Lemoine et al., 1998)* from 201 to 360 were utilized for computing the least-squares modification parameters and long wavelength structure of the quasi-geoid. The integration cap size of the Stokes integral was optimally selected to be 3° by comparing the results with GNSS/levelling data. At the end 19 mm standard error was estimated for the KTH08 in a 4-parameter corrective surface fit to the GNSS/levelling, which is quite satisfying for the present geodetic applications.

5.1.2. The Swedish normal heights

The normal heights over Sweden have been measured using precise motorized levelling since 1979 to 2003; see Fig. 1 *(Eriksson et al., 2002)*. The motorized levelling is an efficient method to measure the height difference as quickly as possible with preserving the quality of the measurements. About 50000 benchmarks have been created during this levelling mission; see Fig. 2 *(Eriksson et al., 2002)*. The quality of the heights is more or less similar with an error of 5–10 mm (*Ågren, 2009*, personal communication). Unlike the common belief of creating zero-, first- and second order levelling national networks, there is not such a classification of heights in Sweden. It means that the levelling network of Sweden was not classified at all. Computations of the heights were carried out in cooperation with the other Nordic countries and finished in 2005. All the normal heights are referenced with respect to Normaal Amsterdam Peil as the zero level. In these computations the land uplift model NKG2005LU *(Ågren and Svensson, 2006)* was utilized.

Fig. 1. Motorized levelling *(Eriksson et al., 2002)*

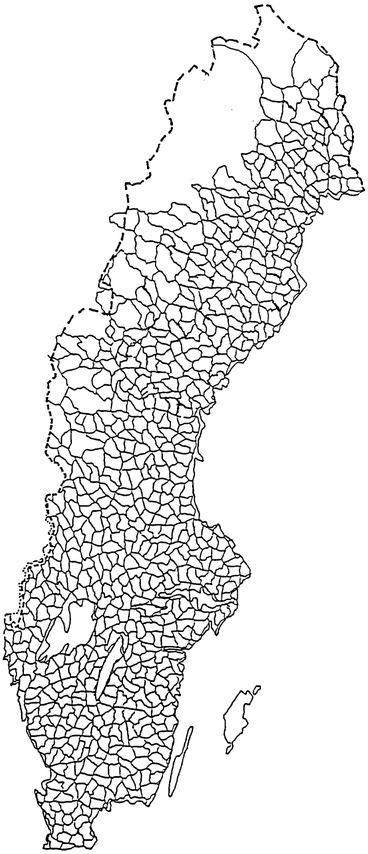


Fig. 2. Normal heights network *(Eriksson et al., 2002)*

5.1.3. The Swedish GNSS heights

Unlike the normal heights, the Swedish GNSS heights were measured in three steps. At the first step, a zero-order network with 5–10 mm error was established. This network consists of 25 permanent GNSS stations whose coordinates were defined in the Swedish reference system 99. The height of 181 points was determined relative to the zero-order network using 48 hours of observations with digital multimedia technologies antennas and the Bernese software in the first-order network.

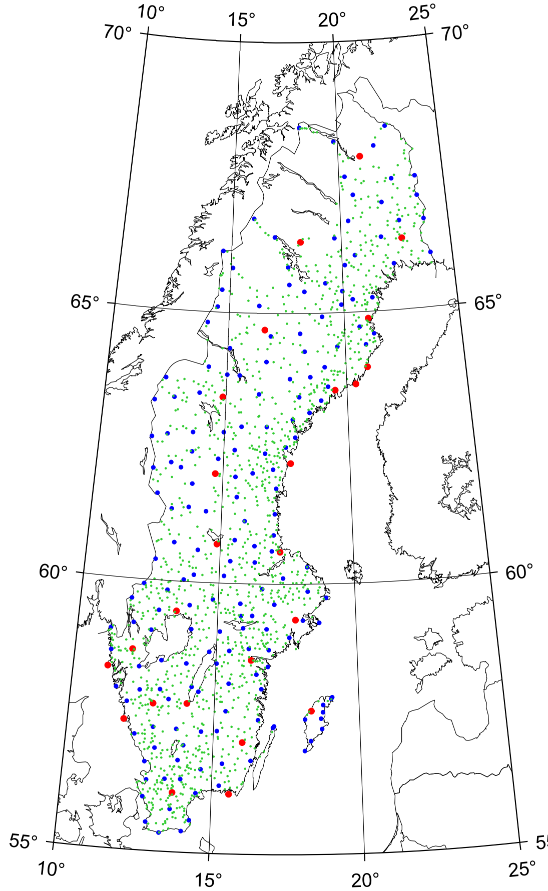


Fig. 3. Distribution of GNSS/levelling points *(Ågren, 2009)*

According to the utilized instruments and duration of the observations, an error of 10–20 mm is expected for these GNSS heights. The second-order network of GNSS heights consists of 1364 points which were established based on densification of the previous networks using static GNSS positioning with 0.5–1 hours of observations. An error of 15–30 mm is expected to the GNSS heights in this network. In total 1570 GNSS levelling points were established over Sweden (Fig. 3) which is extremely good for calibration of the KTH08 quasi-geoid model of Sweden.

5.2. Numerical investigations in VCE based on different SMs

The results of the combined adjustment of the GNSS/levelling data and the quasi-geoid over Sweden are presented in Tab. 1. The table shows the statistics of the misclosures and the residuals after the adjustment. The 7-parameter surface, Eq. (5c), is most convenient to present the discrepancies between the KTH08 quasi-geoid model and the levelling data.

Table 1. Statistics of misclosures and residuals after fitting GNSS/levelling data to KTH08 quasi-geoid model. Unit: 1 mm

	Max.	Mean	Min.	Std.
before fitting	754	669	598	24
4-parameter, Eq. (5a)	82	0	−89	21
5-parameter, Eq. (5b)	82	0	−86	21
7-parameter, Eq. (5c)	85	0	−80	19

In this combined adjustment process, the variance–covariance matrix of the observations was diagonal and the diagonal elements were selected according to the error of the heights and the quasi-geoid presented in Subsection 5.1. Fig. 4 illustrates the misclosures and the residuals after fitting the 7-parameter corrective surface. Since the plots of other surfaces were qualitatively similar we present the fitting residuals of the 7-parameter surface only. Figures (5a), (5b) and (5c) illustrate the three-dimensional plots

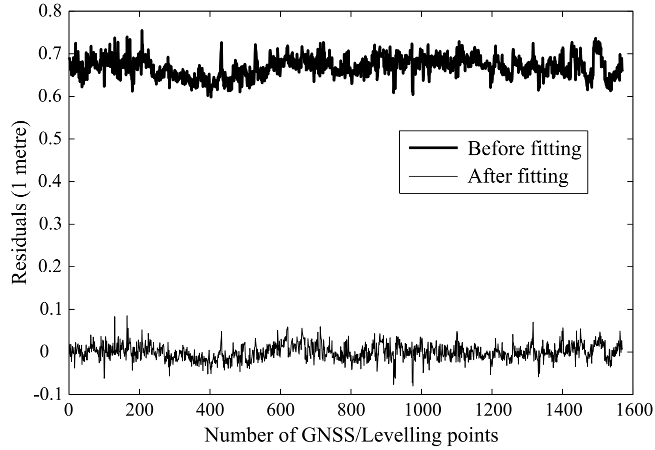


Fig. 4. Misclosues and fitting residuals of 7-paramter corrective surface.

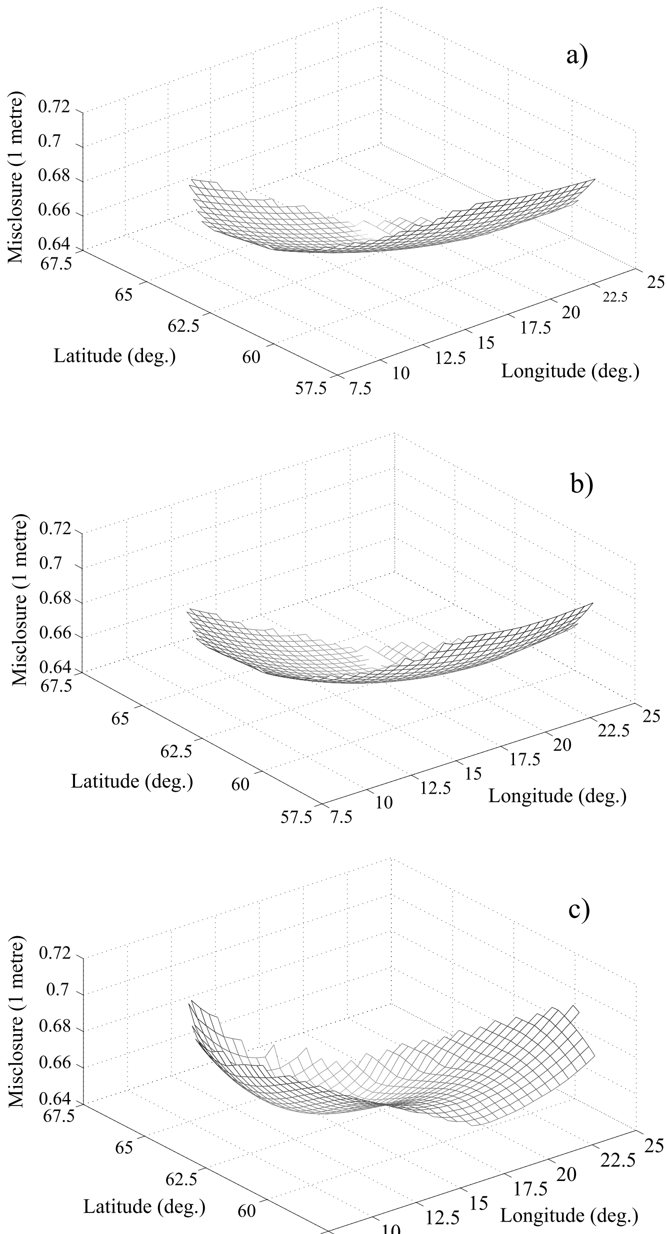


Fig. 5. Corrective surfaces a) 4-parameter, b) 5-parameter and c) 7-parameter.

of 4-, 5- and 7-parameters corrective surfaces, respectively.

5.2.1. Numerical investigations in VCE based on the two-VC SM

Since the geometric quasi-geoid is given by $\zeta_1 = h_{\mathrm{GNSS}} - h_{\mathrm{Normal}}$, using the error propagation law and supposing that the observables are uncorrelated, the standard error of ζ_1 will take the form:

$$\sigma_1 = \sqrt{\sigma^2_{h_{\mathrm{GNSS}}} + \sigma^2_{h_{\mathrm{Normal}}}}. \tag{12}$$

As noted above, $\sigma_{h_{\mathrm{Normal}}} = 10$ mm and $\sigma_{h_{\mathrm{GNSS}}}$ is 10, 20 and 30 mm in zero-, first and second-order GNSS heights networks, respectively. Consequently, based on Eq. (12) the propagated error of ζ_1 will be 14, 22 and 32 mm, depending on the order of the networks. Figure 6 shows the VC ratios (ratios of VCs in two frequent iterations) of ζ_1 and ζ_2 during iteration based on a) 4-, b) 5- and 7-parameter corrective surface models. The figure illustrates that the ratios converge to 1 in 6 iterations in a convergence level of 0.0001. It means that the difference between each VC in the two last iterations is smaller than 0.0001. This value is the convergence criterion in all the numerical computations in Subsection 5.2. The value of the VCs σ_1^2 and σ_2^2 and the estimated errors $\bar{\sigma}_1$ and $\bar{\sigma}_2$ of ζ_1 and ζ_2, which are the average value of their errors, are presented in Table 2.

Here we denote the summation of the normal heights and quasi-geoid by $\hat{h}_{\mathrm{GNSS}}$:

$$\hat{h}_{\mathrm{GNSS}} = h_{\mathrm{Normal}} + \zeta. \tag{13a}$$

Now the SM is considered in such a way that the errors of the GNSS heights and $\hat{h}_{\mathrm{GNSS}}$ are calibrated. In this case the error of $\hat{h}_{\mathrm{GNSS}}$ can be estimated through the error propagation law. Since the error of the quasi-geoid is about 19 mm and the error of the normal heights 10 mm, the error for $\hat{h}_{\mathrm{GNSS}}$ is

$$\sigma_1 = \sqrt{\sigma^2_{\zeta} + \sigma^2_{h_{\mathrm{Normal}}}} = 21 \text{ mm}. \tag{13b}$$

Considering σ_1^2 as the VC of h_{GNSS} and σ_2^2 for $\hat{h}_{\mathrm{GNSS}}$, the VCE process is performed. The VC ratios are presented in Fig. 7, showing a similar pattern of the ratios as that presented in Fig. 6. Table 3 illustrates the values of

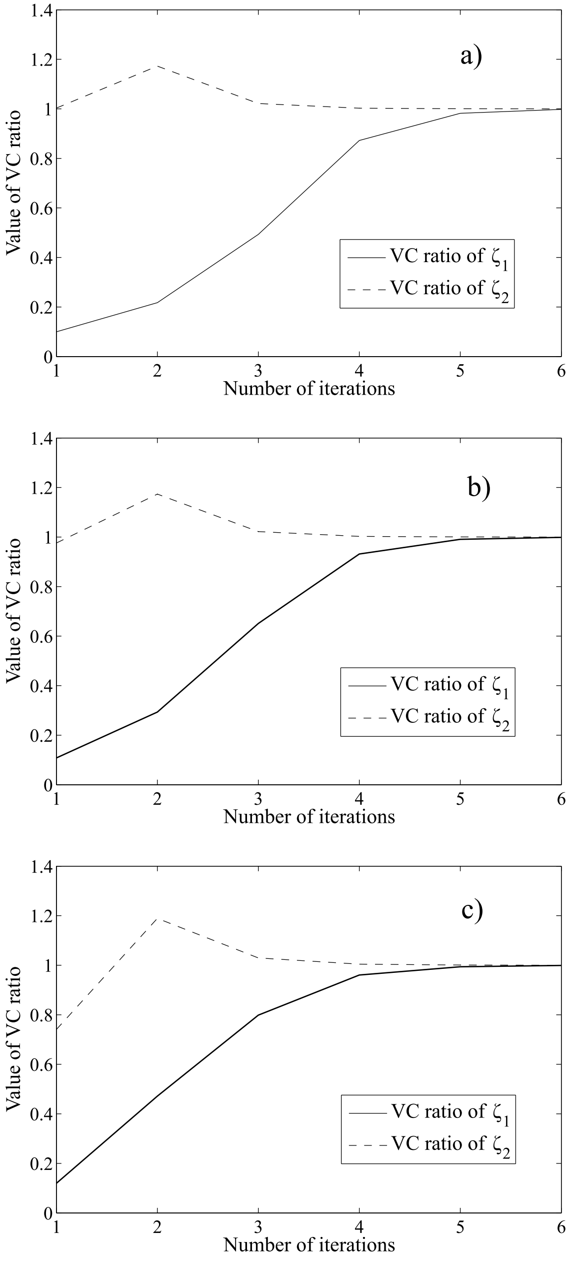


Fig. 6. VC ratios of ζ_1 and ζ_2 during iteration for a) 4-, b) 5- and c) 7-parameter corrective surfaces (initial values of all VCs are equal to 1).

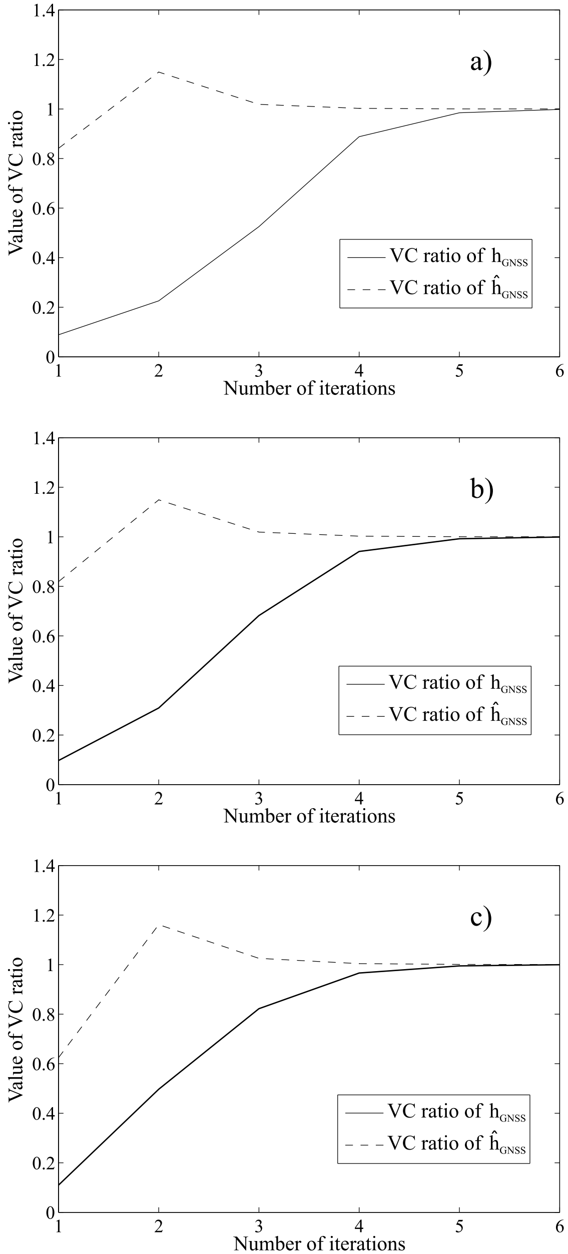


Fig. 7. VC ratios of h_{GNSS} and $\hat{h}_{\mathrm{GNSS}}$ during iteration for a) 4-, b) 5- and c) 7-parameter corrective surface (initial values of all VCs are equal to 1).

Table 2. Values of VCs and estimated errors of ζ_1 ($\bar{\sigma}_1$) and ζ_2 ($\bar{\sigma}_2$)

	parameter	4	5	7
VCs	σ_1^2	0.00	0.01	0.04
	σ_2^2	1.21	1.18	0.93
Error (1 mm)	$\bar{\sigma}_1$	± 2	± 4	± 6
	$\bar{\sigma}_2$	± 21	± 21	± 18

Table 3. Values of VCs and estimated errors of h_{GNSS} ($\bar{\sigma}_1$) and $\hat{h}_{\text{GNSS}}$ ($\bar{\sigma}_2$)

	parameter	4	5	7
VCs	σ_1^2	0.01	0.02	0.04
	σ_2^2	0.97	0.96	0.75
Error (1 mm)	$\bar{\sigma}_1$	± 3	± 4	± 6
	$\bar{\sigma}_2$	± 21	± 21	± 18

VCs and the estimated errors of h_{GNSS} $\hat{h}_{\text{GNSS}}$.

Comparing Tables 2 and 3 we can see that the VCs of ζ_1 and h_{GNSS} are considerably smaller than those of ζ_2 and $\hat{h}_{\text{GNSS}}$. It means that their a-priori errors are very large so that, in order to balance the errors in accordance with misclosures, they have to be multiplied by small VCs. Both ζ_1 and h_{GNSS} consist of very precise normal heights having the accuracy of 10 mm over the country. Unlike σ_1^2, σ_2^2 has a value close to 1, indicating that the a-priori error of ζ_2 was reasonably selected. Correspondingly, one can expect smaller estimated error for ζ_1 and h_{GNSS} than ζ_2 and $\hat{h}_{\text{GNSS}}$.

5.2.2. Numerical investigations on VCE based on the three-VC SM

In the theorem presented in Section 3, when the cofactor matrices are equal to an identity matrix the VCs are not estimable as system of Eq. (4a) will have a rank of 1 and it will be singular. However, in this study we have different errors for the GNSS heights, one error for all the normal heights and one for the quasi-geoid. This matter makes two rows and two columns of the matrix $\mathbf{S}$ similar and the rank of the system of equations is reduced by 2 and $\mathbf{S}$ becomes singular. The main reason of this singularity is due to selecting an identity matrix for $\mathbf{Q}_2$ and $\mathbf{Q}_3$, or it is better to say that, they have the same diagonal elements. We already know that the error of the normal heights is 10 mm and it is 19 mm for the quasi-geoid. Therefore it will not be unreasonable to randomly vary the diagonal elements of $\mathbf{Q}_2$ and $\mathbf{Q}_3$ so that these matrices have different values for their diagonal elements. This process can be done by randomizing the diagonal element based on a normal distribution with zero mean and a standard deviation equal to that was claimed for the observations. In this case, system of Eqs. (4a) will not be singular and the VCs are uniquely estimable. One can consider many realizations of these random variations for the diagonal elements. We know that the improper choice of the a-priori errors or their inconsistency with the misclosures is one of the reasons of coming out negative VCs. However, our aim is to vary the diagonal elements to get out of the singularity problem. Consequently, we have the right of selecting that realization which is consistent with the misclosures. In such case no negative VCs is come out. In order to make sure that the results are reliable several non-negative realizations can be considered and tested and the mean value of the estimated error can be selected as the calibrated ones.

Figure 8 shows the VC ratios of h_{GNSS}, h_{Normal} and ζ during the VCE process. Figs. (8a), (8b) and (8c) represent the VC ratios based on 4-, 5- and 7-parameter models, respectively. The VC ratios are very similar when the 4- and 5-parameter models are used but they differ for the 7-parameter model. Figure 8 shows that all the VC ratios converge to 1 in 7 iterations (at a convergence level of 0.0001).

Table 4 shows the values of the VCs and the estimated errors of h_{GNSS}, h_{Normal} and ζ. The table shows that the VCs are more or less the same

Table 4. Value of VCs and estimated error of h_{GNSS} ($\bar{\sigma}_1$), h_{Normal} ($\bar{\sigma}_2$) and ζ ($\bar{\sigma}_3$)

	VCs			Error (1 mm)		
	σ_1^2	σ_2^2	σ_3^2	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{\sigma}_3$
4-parameter	0.52	0.14	0.11	± 21	± 4	± 7
5-parameter	0.50	0.14	0.11	± 21	± 4	± 6
7-parameter	0.40	0.24	0.11	± 18	± 5	± 6

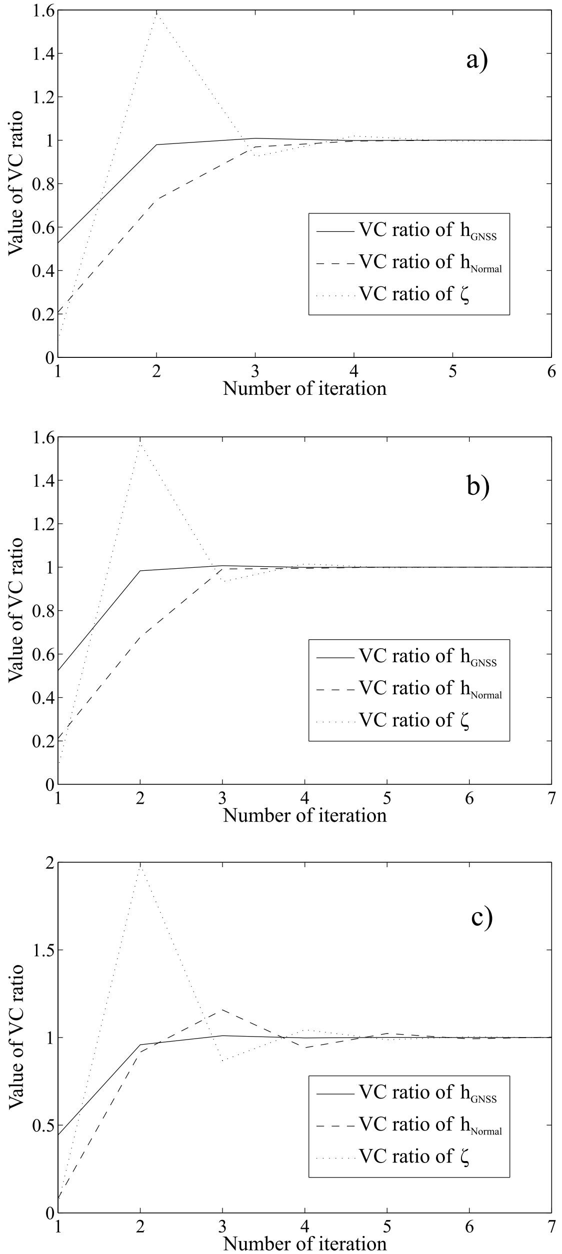


Fig. 8. VC ratios of h_{GNSS}, h_{Normal} and ζ during iteration for a) 4-, b) 5- and c) 7-parameter corrective surfaces (initial values of all VCs are equal to 1).

for the 4- and 5-parameter models. We can expect that their error estimates will be in the same order as well. As we saw, the 7-parameter model can describe the biases and the tilts of the quasi-geoid well. We observe different values for the VCs and the estimated errors based on this model. However, the changes in the magnitude of the VCs insignificantly change the estimated errors.

5.2.3. Numerical investigations in VCE based on the four-VC SM

In Fig. 9 $h1_{\mathrm{GNSS}}$, $h2_{\mathrm{GNSS}}$ and $h3_{\mathrm{GNSS}}$ stand for the GNSS heights of the zero-, first- and second-order networks. $\hat{h}_{\mathrm{GNSS}}$ is the same with that was defined in Eq. (13a). Figures (9a), (9b) and (9c) show the VC ratios of the zero-, first- and second-order networks of the GNSS heights using 4-, 5- and 7-parameter models, respectively. Figures (9a), (9b) and (9c) show that the VC ratios of $h2_{\mathrm{GNSS}}$ and $h3_{\mathrm{GNSS}}$ are closed to 1 in 2 iterations and VC ratio of $h1_{\mathrm{GNSS}}$ in 3. This means that the weights of these heights are balanced faster than that of $\hat{h}_{\mathrm{GNSS}}$. This phenomenon is normal according to the selected SM (10b) as for three cofactor matrices of $\mathbf{Q}_4, \mathbf{Q}_5$ and $\mathbf{Q}_6$ one VC is estimated. The figures show the VC ratios converge to 1 in 7 iterations.

The thing that we learn from Table 5 is that, the estimated error $\bar{\sigma}_3$, which belongs to the second-order GNSS heights, is in the same order as that of the first-order heights. Also we observe that the error of the zero-order GNSS heights $\bar{\sigma}_1$ is about 13 mm and slightly larger than the priori value (10 mm). The table says that the error of $\hat{h}_{\mathrm{GNSS}}$ should be about 3 mm which is smaller than the a-priori value of its error (21 mm). One observes that σ_1^2 differs by the choice of the corrective surface model and the consequence of such a change in the estimated error is about 3 mm.

Table 5. Value of VCs and estimated error of $h1_{\mathrm{GNSS}}$ ($\bar{\sigma}_1$), $h2_{\mathrm{GNSS}}$ ($\bar{\sigma}_2$), $h3_{\mathrm{GNSS}}$ ($\bar{\sigma}_3$) and $\hat{h}_{\mathrm{GNSS}}$ ($\bar{\sigma}_4$)

	VCs				Errors (1 mm)			
	σ_1^2	σ_2^2	σ_3^2	σ_4^2	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{\sigma}_3$	$\bar{\sigma}_4$
4-parameter	2.73	1.17	0.48	0.02	± 16	± 22	± 21	± 3
5-parameter	2.69	1.15	0.48	0.01	± 16	± 22	± 21	± 3
7-parameter	1.83	0.95	0.40	0.03	± 13	± 19	± 19	± 3

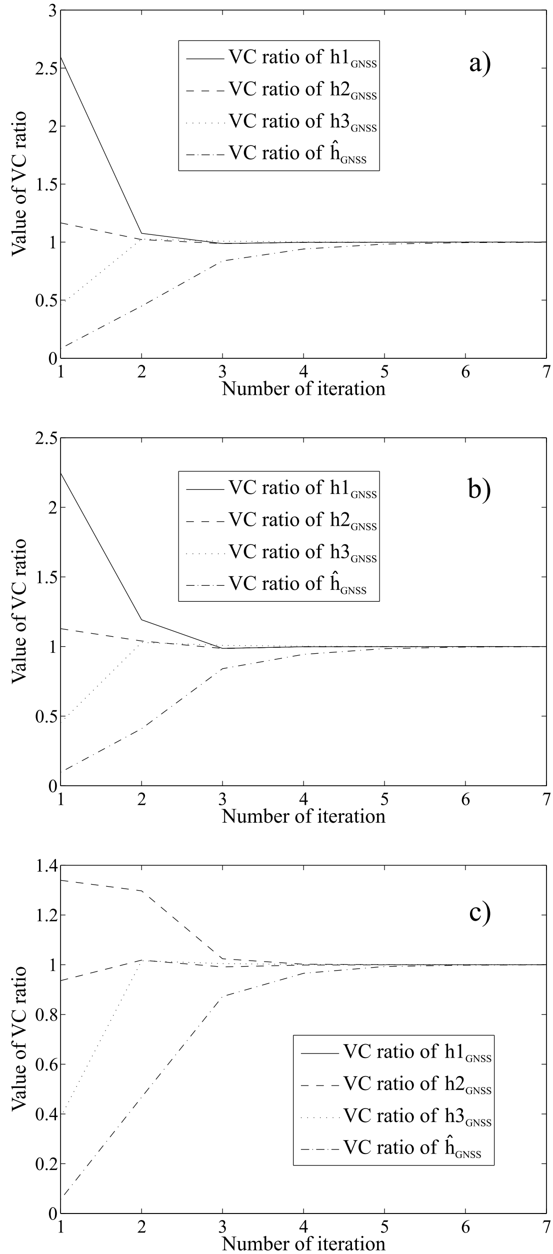


Fig. 9. VC ratios of $h1_{\mathrm{GNSS}}$, $h2_{\mathrm{GNSS}}$, $h3_{\mathrm{GNSS}}$ and $\hat{h}_{\mathrm{GNSS}}$ during iteration for a) 4-, b) 5- and c) 7-parameter corrective surfaces (initial values of all VCs are equal to 1).

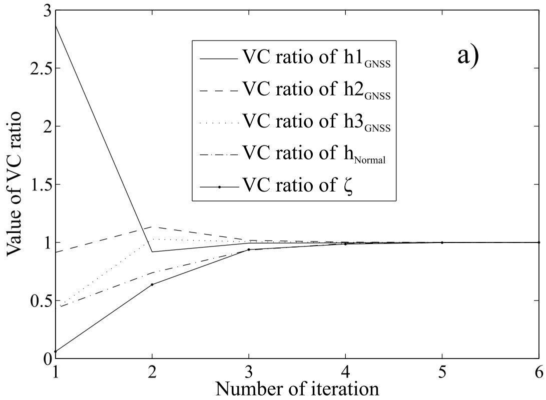
5.2.4. Numerical investigations in VCE based on the five-VC SM

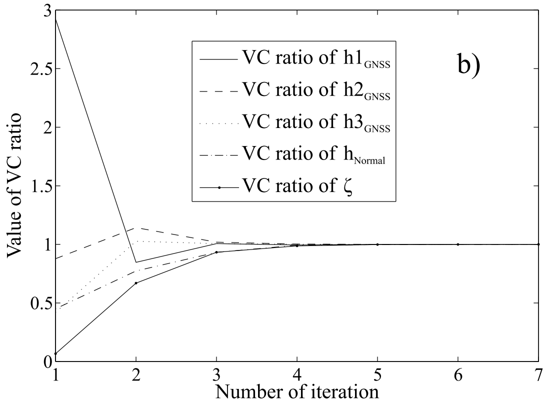
Figures (10a), (10b) and (10c) show the VC ratios based on fitting a 4-, 5-, and 7-parameter model, respectively to the misclosures of the GNSS/levelling points. Figures (10a) and (10b) are very similar, which shows that the difference between the results of VCE using 4- and 5-parameters models is insignificant. Figure 10 illustrates that all the VC ratios converge to 1 in 6 iterations.

Table 6 presents the values of the VCs and the estimated errors of $h1_{\mathrm{GNSS}}$, $h2_{\mathrm{GNSS}}$ and $h3_{\mathrm{GNSS}}$, h_{Normal} and ζ. As we see σ_1^2, which is related to $h1_{\mathrm{GNSS}}$ (the zero-order GNSS heights), is more or less the same as that is for the 4- and 5-parameter models. The estimated error for these points is about 16 mm, which is not consistent with our a-priori assumption for their errors (5–10 mm). $\bar{\sigma}_2$ is in agreement with the presumed error (10–20 mm) for $h2_{\mathrm{GNSS}}$. However, the estimated error of $h3_{\mathrm{GNSS}}$ seems to be optimistic, as its a-priori error was 15–30 mm. The table says that the error of h_{Normal} and ζ is 3 mm and better than the presumed one.

The idea of VCE in an adjustment process is to balance the a-priori errors of the observables with the residuals or misclosures estimated after the adjustment procedure. When the misclosures or the residuals are small it is expected that the VC are come out small to scale the priori errors with the magnitude of the misclosures and vice versa. As Table 1 shows the standard deviation of the misclosures before fitting the corrective surfaces is about 24 mm and quite small. However, we have assumed an error of 19 mm for the quasi-geoid, 10 mm for the normal heights and 10, 20 and 30 mm for the GNSS heights which are not consistent with the standard deviation of the misclosures before and even after the fittings. Therefore observing small estimated errors for the heights after the VCE process is normal.

Tables 7, 8 and 9 show the estimated parameter of 4-, 5- and 7-parameter corrective surfaces and their errors. The numbers in the SM columns means the type of SM which is used in VCE. It can be two-, three-, four- and five-VC SM corresponding to 2, 3, 4 and 5, respectively. Table 8 shows a large error for x_0 which is more or less in the same level of the value of x_0. It means that adding one parameter to a corrective surface of 4-parameter will not improve the combined adjustment with VCE and this parameter is not significant in practice. This also true by observing very similar results





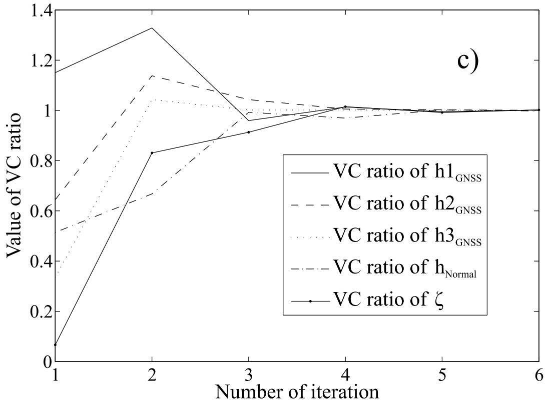


Fig. 10. VC ratios of $h1_{\mathrm{GNSS}}$, $h2_{\mathrm{GNSS}}$ and $h3_{\mathrm{GNSS}}$, h_{Normal} and ζ during iteration for a) 4-, b) 5- and c) 7-parameter corrective surfaces (initial values of all VCs are equal to 1).

based on fitting 4- and 5-parameter correctives surfaces.

Table 6. Value of VCs and estimated error of $h1_{\mathrm{GNSS}}$ ($\bar{\sigma}_1$), $h2_{\mathrm{GNSS}}$ ($\bar{\sigma}_2$), $h3_{\mathrm{GNSS}}$ ($\bar{\sigma}_3$), h_{Normal} ($\bar{\sigma}_4$) and ζ ($\bar{\sigma}_5$)

	VCs					Errors (1 mm)				
	σ_1^2	σ_2^2	σ_3^2	σ_4^2	σ_5^2	$\bar{\sigma}_1$	$\bar{\sigma}_2$	$\bar{\sigma}_3$	$\bar{\sigma}_4$	$\bar{\sigma}_5$
4-parameter	2.61	1.06	0.44	0.33	0.03	± 16	± 21	± 20	± 5	± 4
5-parameter	2.48	1.03	0.43	0.32	0.04	± 16	± 21	± 20	± 6	± 4
7-parameter	1.48	0.77	0.35	0.29	0.05	± 12	± 18	± 18	± 6	± 4

Table 7. Estimated parameters of 4-parameter corrective surface for different SMs

SM	x_0	x_1	x_2	x_3
2	6.04 ± 0.30	−2.47 ± 0.14	−1.20 ± 0.06	−4.65 ± 0.26
3	5.60 ± 0.30	−2.26 ± 0.13	−1.10 ± 0.06	−4.28 ± 0.26
4	6.03 ± 0.30	−2.47 ± 0.14	−1.21 ± 0.06	−4.64 ± 0.26
5	6.02 ± 0.30	−2.47 ± 0.14	−1.20 ± 0.06	−4.64 ± 0.26

Table 8. Estimated parameters of 5-parameter corrective surface for different SMs

SM	x_0	x_1	x_2	x_3	x_4
2	1.66 ± 1.32	−4.68 ± 0.67	−1.84 ± 0.20	12.12 ± 4.96	−12.00 ± 3.54
3	1.65 ± 1.31	−4.47 ± 0.65	−1.76 ± 0.19	11.52 ± 4.88	−11.44 ± 3.49
4	1.59 ± 1.32	−4.72 ± 0.66	−1.86 ± 0.20	12.39 ± 4.94	−12.20 ± 3.53
5	1.61 ± 1.31	−4.71 ± 0.66	−1.86 ± 0.19	12.35 ± 4.92	−12.16 ± 3.52

Table 9. Estimated parameters of 7-parameter corrective surface for different SMs

SM	x_0	x_1	x_2	x_3	x_4	x_5	x_6
2	−127.97 ± 8.30	−33.07 ± 1.79	−367.95 ± 23.14	124.31 ± 8.03	31.04 ± 1.89	153.65 ± 10.07	214.25 ± 13.23
3	−129.01 ± 7.77	−31.40 ± 1.73	−366.09 ± 22.07	124.91 ± 7.55	28.98 ± 1.83	152.03 ± 9.69	214.12 ± 12.53
4	−127.93 ± 8.00	−33.04 ± 1.78	−367.95 ± 23.00	124.28 ± 8.02	31.00 ± 1.89	153.69 ± 10.04	214.21 ± 13.21
5	−127.79 ± 8.28	−32.84 ± 1.78	−366.62 ± 23.07	124.05 ± 8.01	30.74 ± 1.88	152.94 ± 10.04	213.65 ± 13.20

6. Conclusions

In this study we calibrated the errors of the GNSS and the normal heights as well as the quasi-geoid KTH08 model over Sweden through a VCE process in a combined adjustment model. The 7-parameter corrective surface yields the best fit to the misclosures of the GNSS/levelling points and the KTH08 model with a standard deviation of 19 mm. Different SMs were used to estimate the VCs. When the three-VC model was used the calibrated error of the GNSS heights became 18 mm, and the errors of the normal heights and the KTH08 model are 5 and 6 mm, respectively. In the case where we estimate one VC for each order of the GNSS heights, we estimated errors of 13, 19 and 19 mm for the zero-, first- and the second-order GNSS heights, respectively. In this case, the error of the reconstructed GNSS heights from the quasi-geoid and levelling heights was 3 mm. In the case of using a 5-component stochastic model in which five VCs are estimated, errors of 12, 19 and 18 mm were estimated for the zero-, first and the second-order GNSS heights, 3 mm for the normal heights and the KTH08 quasi-geoid model.

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