Refraction effect in the heat flow due to a 3-D prismoid, situated in a two-layered earth

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Abstract: We present a mathematical model for the disturbance of the stationary geothermal field due to a three-dimensional perturbing body embedded near the surface of a two-layered earth. The theoretical analysis is based on the generalized theory of the double-layer potential, similar to the boundary integral method used in the direct current geoelectricity problems. Special attention is paid to the quadrilateral prismoids bounded by planar skew faces. The numerical calculations were performed for a 3-D prismoids (blocks) with thermal conductivity greater or lower than in the ambient layer. Numerous graphs are shown for the disturbance of the heat flow on the surface of the Earth.

Key words: geothermics, heat flow refraction, double-layer potential, boundary integral equations, boundary element methods, solid angle calculations

1. Introduction

The heat flow from the Earth's interior is of interest in geothermal prospecting based on geothermal models (e.g., *Chen and Beck, 1991*). The refraction effect in geothermics occurs due to the presence of a 3-D or 2-D perturbing body of different thermal conductivity λ_T with respect to the 'normal' surrounding horizontally - layered medium of thermal conductivity λ_1 if the body is embedded in the 1st layer, $0 < z < h$, or if the body is embedded in the substratum $z > h$ of thermal conductivity λ_2 (Fig. 1). The paper by *Hvoždara and Valkovič (1999)* solved this problem for the rectangular prism.

Physical qualitative analysis clearly indicates that a well-conducting body ($\lambda_T > \lambda_i, i = 1, 2$) attracts heat flux lines to its interior, while a poorly conducting body ($\lambda_T < \lambda_i$) deflects heat flow lines away. It means that we can

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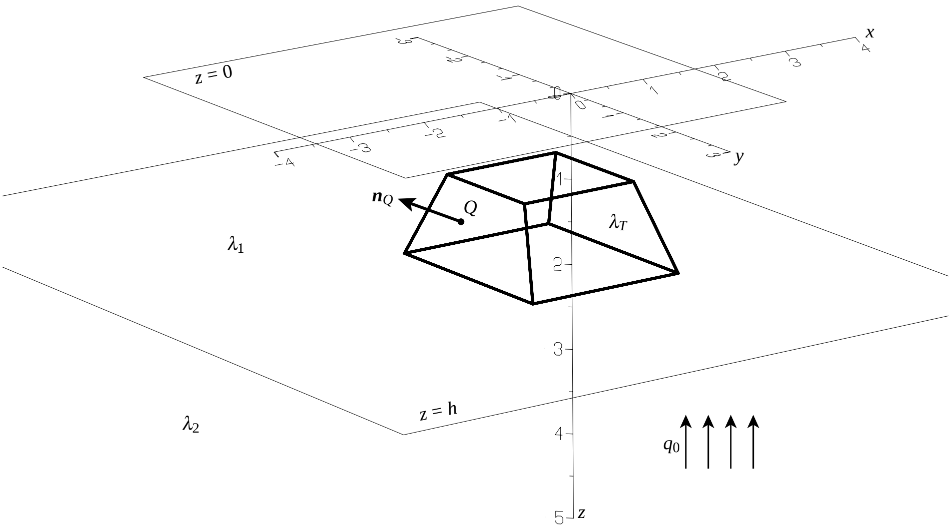


Fig. 1. The 3-D perturbing prismoid situated in a two-layered medium.

expect a positive heat flow anomaly on the surface above the well-conducting body, whereas in the second case a negative heat flow anomaly is expected. Note that the perturbation of the heat flow due to some possible additional heat sources in the anomalous body is not a subject of this paper although in nature such combined effects have been studied, e.g., in *Ljubimova et al. (1983)*.

Some analytical mathematical models of this effect exist, e.g., *Carslaw and Jaeger (1959)*. The 2-D finite difference method has also been applied, e.g., *Majcin (1988)*. For 2-D disturbing bodies, embedded in the halfspace, the boundary element method (BEM) has proved to be an effective tool for numerical modelling, e.g., *Chen and Beck (1991)*. In this paper we present the BEM theory applied on model situations with prismoid in two layered medium and numerical calculations for this perturbing body, quadrilateral prismoid with upper and bottom faces parallel with planes $z = 0, h$.

2. Theoretical background

The theoretical formulation is similar to that in our previous papers *Hvoždara (1982* and *2007)*, which solved mathematically similar potential problems of geoelectricity and for the geothermal problem in *Hvoždara and Valkovič (1999).*

The unperturbed stationary temperature field, linearly dependent on the depth z only, is denoted as $T_1(z)$ for $z \in \langle 0, h \rangle$ and $T_2(z)$ for $z > h$. A simple check shows that the formulae for $T_1(z)$ and $T_2(z)$ are:

$$T_1(z) = q_0 z/\lambda_1, \quad z \in \langle 0, h \rangle, \tag{1}$$

$$T_2(z) = q_0(z - h)/\lambda_2 + q_0 h/\lambda_1, \quad z > h, \tag{2}$$

where q_0 is the unperturbed heat flow density. These functions obey the Laplace equation and continuity of the temperature and heat flow $\lambda \partial T/\partial z$ at the boundary $z = h$.

Due to the presence of perturbing body τ the temperature fields are changed both in layer '1' and substratum '2' by anomalous temperatures $U_1^*(x, y, z)$ and $U_2^*(x, y, z)$. The total temperature fields are:

$$U_1(P) = T_1(P) + U_1^*(P), \tag{3}$$

$$U_2(P) = T_2(P) + U_2^*(P), \tag{4}$$

where $P \equiv (x, y, z)$ is the calculation point. The perturbation parts of $U_1(P)$ and $U_2(P)$ obey Laplace's equation

$$\nabla^2 U_{1,2}^*(P) = 0, \tag{5}$$

with zero limit for $P \to \infty$ in all directions from perturbing body. The theoretical analysis of the problem shows that we have to find the regular solution of the boundary value problem for Laplace's equation in media '1', '2' and in perturbing body τ, where the temperature field is denoted as $U_T(P)$:

$$\nabla^2 U_1^*(P) = 0, \qquad \nabla^2 U_2^*(P) = 0, \nabla^2 U_T^*(P) = 0, \tag{6a,b}$$

$$\lim_{P\to\infty} U_1^* = 0, \qquad \lim_{P\to\infty} U_2^* = 0, \tag{7a,b}$$

$$|U_T(P)| < +\infty, \quad P \in \tau$$

$$U_1(P)|_{z=0} = 0, \tag{8}$$

$$U_1(P)|_{z=h} = U_2(P)|_{z=h}\,, \qquad \lambda_1 \left.\frac{\partial U_1(P)}{\partial z}\right|_{z=h} = \lambda_2 \left.\frac{\partial U_2(P)}{\partial z}\right|_{z=h}, \tag{9,10}$$

$$U_1(P)|_{S} = U_T(P)|_{S}\,, \qquad \lambda_1 \left.\frac{\partial U_1(P)}{\partial n}\right|_{S} = \lambda_T \left.\frac{\partial U_T(P)}{\partial n}\right|_{S}. \tag{11,12}$$

Here $\partial .../\partial n$ denotes the derivative with respect to the outer normal $\boldsymbol{n}$ to the surface S of the 3-D body τ. This potential problem is mathematically similar to the geoelectrical problems solved earlier by *Hvoždara (1982, 1995)*. The only principal difference is in the boundary condition (8) which says that the temperature on the surface of the Earth is isothermal; this constant temperature can be taken as zero on our (auxiliary) temperature scale. This is expressed by the formulae (1) and (8).

Using an apparatus very similar to the geoelectrical problem mentioned (by means of Green's boundary integral equations in complex media) it can be proved that the solution of our potential problem is the sum of the unperturbed temperatures and boundary integrals representing the perturbation part of the temperature field. Namely

$$U_1(P) = T_1(P) + \frac{1}{4\pi}\int\limits_S f(Q)\frac{\partial}{\partial n_Q}G_1(P,Q)\,\mathrm{d}\,S_Q, \tag{13}$$

$$U_2(P) = T_2(P) + \frac{1}{4\pi}\int\limits_S f(Q)\frac{\partial}{\partial n_Q}G_2(P,Q)\,\mathrm{d}\,S_Q, \tag{14}$$

$$U_T(P) = \frac{\lambda_1}{\lambda_T}\left[T_1(P) - v_0 + \frac{1}{4\pi}\int\limits_S f(Q)\frac{\partial}{\partial n_Q}G_2(P,Q)\,\mathrm{d}\,S_Q\right] + v_0, \tag{15}$$

where $G_1(P,Q)$ and $G_2(P,Q)$ are Green's functions and $\partial G_{1,2}(P,Q)/\partial n_Q$ denote their derivatives with respect to the outer normal $\boldsymbol{n}_Q$ on the surface of the perturbing body. The surface S of the perturbing body is assumed to be piecewise smooth in Lyapunov's sense. Function $f(Q)$ expresses the distribution of the double-layer density distributed on the surface S, it must be determined by solving the boundary integral equation as will be shown next. Point $Q \equiv (x', y', z')$ is the moving point shifted along S. Constant v_0 is the mean value of the unperturbed temperature $T_1(P)$ on the surface S:

$$v_0 = \frac{1}{S} \int\limits_{s} T_1(P) \, \mathrm{d}\, S_P. \tag{16}$$

The double layer density $f(Q)$ must be determined by solving the boundary integral equation as will be shown next.

3. The Green's functions and the boundary integral equation (B.I.E.) solution

Boundary conditions (7a,b) and (8)–(10) can be fulfilled by the proper determination of Green's functions. These Green's functions G_1, G_2 must obey 3-D partial Poisson's or Laplace's equation:

$$\nabla^2 G_1(P,Q) = -4\pi\delta(P,Q), \quad z \in \langle 0, h \rangle \tag{17}$$

and

$$\nabla^2 G_2(P,Q) = 0, \quad z > h, \tag{18}$$

where $\delta(P,Q)$ is a 3-D Dirac function whose pole is at point $Q \equiv (x', y', z') \in \tau$. These Green's functions satisfy similar boundary conditions on $z = 0$, and $z = h$:

$$G_1(P,Q)|_{z=0} = 0 \tag{19}$$

$$[G_1(P,Q) - G_2(P,Q)]|_{z=0} = 0 \tag{20a}$$

$$[\lambda_1 \partial G_1(P,Q)/\partial z]|_{z=h} = [\lambda_2 \partial G_2(P,Q)/\partial z \,]|_{z=h}\,, \tag{20b}$$

$$\lim_{P \to \infty} G_{1,2}(P,Q) = 0. \tag{21}$$

Physically $G_1(P,Q)$ resp. $G_2(P,Q)$ represent the temperature of the point heat source located at point Q and calculated for point P in the upper layer ($G_1(P,Q)$) or in substratum ($G_2(P,Q)$). But the common source multiplicator $q_0/(4\pi\lambda_1)$ is replaced by 1 in order to satisfy Poisson's equation (17), which has the non-trivial solution:

$$g_1(P,Q) = 1/R = \left[(x - x')^2 + (y - y')^2 + (z - z')^2 \right]^{-1/2}, \tag{22}$$

since $\nabla^2 g_1(P,Q) = -4\pi\delta(P,Q)$. The remaining part of $G_1(P,Q)$, i.e. $\tilde{G}_1(P,Q)$ satisfies Laplace's equation.

Let us choose an auxiliary cylindrical system (r,φ,z) whose polar axis runs through point Q perpendicularly to boundaries $z = 0$ and $z = h$. Green's functions will then be independent of azimuthal angle φ and Laplace's equation for harmonic functions $\tilde{G}_1(P,Q)$ and $\tilde{G}_2(P,Q)$ takes the form:

$$\frac{\partial^2 \tilde{G}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{G}}{\partial r} + \frac{\partial^2 \tilde{G}}{\partial z} = 0. \tag{23}$$

The particular solution can be found by applying the method of separation of variables:

$$\tilde{G}(r,z) = J_0(tr)\begin{cases} \mathrm{e}^{tz} \\ \mathrm{e}^{-tz} \end{cases}, \tag{24}$$

where $J_0(tr)$ is the Bessel function of the 1st kind and zero order. $G_1(r,z)$ and $G_2(r,z)$ can be expressed as:

$$G_1(r,z) = \left[r^2 + (z-z')^2\right]^{-1/2} - \left[r^2 + (z+z')^2\right]^{-1/2} + \\ + \int_0^\infty A\left[\mathrm{e}^{-tz} - \mathrm{e}^{+tz}\right] J_0(tr)\,\mathrm{d}\,t \tag{25}$$

where $[r^2 + (z+z')^2]^{-1/2} = 1/R_+$ is the mirroring term to $1/R$ and

$$G_2(r,z) = \int_0^\infty B\,\mathrm{e}^{-tz} J_0(tr)\,\mathrm{d}\,t. \tag{26}$$

It is clear that the radial coordinate r is expressed in the original Cartesian co-ordinate system as:

$$r = \left[(x-x')^2 + (y-y')^2\right]^{1/2}. \tag{27}$$

It can be easily verified that Green's function $G_1(P,Q)$ obeys the boundary equation (19) at surface $z = 0$. In order to find functions A and B, we express $1/R$ and $1/R_+$ in terms of the Weber–Lipschitz integral:

$$\frac{1}{R} = \int_0^\infty \mathrm{e}^{-t|z-z'|} J_0(tr)\,\mathrm{d}\,t, \quad \frac{1}{R_+} = \int_0^\infty \mathrm{e}^{-t(z+z')} J_0(tr)\,\mathrm{d}\,t. \tag{28}$$

In order to satisfy the boundary conditions at $z = h$ we obtain the system of linear equations for A, B:

$$A\left(\mathrm{e}^{-th} - \mathrm{e}^{th}\right) - B\,\mathrm{e}^{-th} = -\mathrm{e}^{-t(h-z')} + \mathrm{e}^{-t(h+z')},$$

$$-A\left(\mathrm{e}^{-th} + \mathrm{e}^{th}\right) + \frac{\lambda_2}{\lambda_1} B\,\mathrm{e}^{-th} = \mathrm{e}^{-t(h-z')} - \mathrm{e}^{-t(h+z')}. \tag{29}$$

We can easily solve this system:

$$A = -k\left[\mathrm{e}^{-t(2h+z')} - \mathrm{e}^{-t(2h-z')}\right]\left[1 - k\,\mathrm{e}^{-2th}\right]^{-1}, \tag{30}$$

$$B = \mathrm{e}^{+tz'} - \mathrm{e}^{-tz'} - k\left(1 - \mathrm{e}^{2th}\right)\left[\mathrm{e}^{-t(2h+z')} - \mathrm{e}^{-t(2h-z')}\right]\left[1 - k\,\mathrm{e}^{-2th}\right]^{-1}, \tag{31}$$

where

$$k = (1 - \lambda_1/\lambda_2)/(1 + \lambda_1/\lambda_2). \tag{32}$$

Now we can use the well-known expansion of factor $[1 - k\,\mathrm{e}^{-2th}]^{-1}$ into the infinite geometrical series:

$$\left[1 - k\,\mathrm{e}^{-2th}\right]^{-1} = \sum_{n=0}^{\infty} k^n \mathrm{e}^{-t2nh}, \tag{33}$$

since $|k\,\mathrm{e}^{-2th}| < 1$. Using the Weber-Lipschitz integral in (25) and (26) we obtain convenient expressions for $G_1(P,Q)$ and $G_2(P,Q)$:

$$\begin{aligned} G_1(P,Q) = \frac{1}{R} - \frac{1}{R_+} - \sum_{n=1}^{\infty} k^n \Big\{ &\left[r^2 + (2nh - z - z')^2\right]^{-1/2} - \\ &- \left[r^2 + (2nh - z + z')^2\right]^{-1/2} - \\ &- \left[r^2 + (2nh + z - z')^2\right]^{-1/2} + \left[r^2 + (2nh + z + z')^2\right]^{-1/2} \Big\}. \end{aligned} \tag{34}$$

$$G_2(P,Q) = (1-k)\left\{\frac{1}{R} - \frac{1}{R_+} - \sum_{n=1}^{\infty} k^n \left\{\left[r^2 + (2nh + z - z')^2\right]^{-1/2} - \right.\right.$$
$$\left.\left. - \left[r^2 + (2nh + z + z')^2\right]^{-1/2}\right\}\right\}. \tag{35}$$

Note that if $\lambda_2 = \lambda_1$, i.e. $k = 0$, we obtain simple two-term Green's function: $R^{-1} - R_+^{-1}$ for the whole halfspace $z > 0$. Now we can derive the boundary integral equation (B.I.E) for determining the function $f(P)$, which is necessary to calculate the temperatures (13)–(15). Assume that perturbing body τ is not in contact with any of its faces (parts of boundary S) having planar boundaries $z = 0$ or $z = h$. If point P approaches the surface (S) from inside ($P \to S_-$) or from outside ($P \to S_+$), singularities in $G_1(P,Q)$ will occur due to the well-known term R^{-1}. This singularity can be treated using the classical theory of the potential double layer (e.g., *Hvoždara, 1995).* After limit transition applied on (13) yields

$$\lim_{P\to S_+} U_1(P) = T_1(P) + \frac{1}{2}f(P) + \frac{1}{4\pi}\int\!\!\!\!\!\!\diagdown_{S} f(Q)\frac{\partial}{\partial n_Q}G_1(P,Q)\,\mathrm{d}\,S_Q,\ P \in S, \tag{36}$$

where the backward slash on the integral sign denotes integration in the principal sense, i.e. the integration of the part with $\partial R^{-1}/\partial n_Q$ is performed over the whole surface S with the exception of the infinitesimally small area ΔS_p around point $P \in S$, where $\partial R^{-1}/\partial n_Q$ has an integrable singularity. The integration of this singular term on ΔS_p resulted in the contribution $\frac{1}{2}f(P)$ in Eq. (36).

A similar limit transition in (15) from the interior of S reads ($P \to S_-$):

$$\lim_{P\to S_-} U_T(P) =$$
$$= \frac{\lambda_1}{\lambda_T}\left[T_1(P) - v_0 - \frac{1}{2}f(P) + \frac{1}{4\pi}\int\!\!\!\!\!\!\diagdown_{S} f(Q)\frac{\partial}{\partial n_Q}G_1(P,Q)\,\mathrm{d}\,S_Q\right] + v_0. \tag{37}$$

The negative sign of term $\frac{1}{2}f(P)$ is well-known in the theory of the classical double-layer potential as a discontinuity of the double-layer potential on the supporting surface S. According to the boundary condition (11) the r.h.s. of Eqs. (36) and (37) must equal each to other, after some easy algebra we arrive at the B.I.E.:

$$f(P) = 2\beta\,[T_1(P) - v_0] + \frac{\beta}{2\pi}\int\limits_S f(P)\frac{\partial}{\partial n_Q}G_1(P,Q)\,\mathrm{d}\,S_Q,\;\; P \in S \tag{38}$$

where $\beta = (1 - \lambda_T/\lambda_1)/(1 + \lambda_T/\lambda_1)$. Since the normal derivative of the kernel $\partial^2 R^{-1}/\partial n_Q \partial n_p$ is continuous on the supporting surface S we can easily check the validity of boundary condition (12).

The boundary integral solution of our problem is now ready. The B.I.E. (38) can only be solved in some simple cases, but for the block body it must be treated numerically, in analogy (e.g., *Brebbia et al., 1984*) with the approach using the boundary element methods (BEM). In our previous paper *Hvoždara and Valkovič (1999)* we presented modifications for two important situations, namely if the perturbing body is by some part of surface S in contact with the bottom or upper plane boundary of the layer '1'. These analyses are valid also in the present case.

4. Calculation of the solid angle of view for the triangle and quadrangle subarea with general orientation of its normal

In the numerical calculations of B.I.E. there plays fundamental role the calculation of integrals with the kernel of type of the double-layer potential: $\boldsymbol{n}_Q \cdot (\boldsymbol{r} - \boldsymbol{r}')|\boldsymbol{r} - \boldsymbol{r}'|^{-3}$ over a small subsurface ΔF_j which is the part of surface S of the perturbing body Ω_T. In the paper *Ivan (1994)* we can find the explanation for the reliable calculation of such integrals for the triangle planar subarea ΔF_j:

$$\Delta A_j = \int\limits_{\Delta F_j} \frac{\boldsymbol{n}_Q \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3}\,\mathrm{d}\,S_Q = -\Delta\Phi_j \tag{39}$$

where $\Delta\Phi_j$ is the solid angle of view from the point $P(\boldsymbol{r})$ onto the planar triangle subarea ΔF_j with outer normal $\boldsymbol{n}' \equiv (n'_x, n'_y, n'_z) \equiv \boldsymbol{n}_Q$. The formula given by *Ivan (1994)* is:

$$\Delta A_j = 2A_{123} = $$
$$= 2\sum_{1,2,3} \operatorname{arctg} \frac{2w_{12}d_{12}}{(R_1 + R_2 + d_{12})\,|R_1 + R_2 - d_{12}| + 2q(R_1 + R_2)}. \tag{40}$$

Geometrical parameters for the formula (40) for the triangle with vertices T_1, T_2, T_3 are depicted in Fig. 2. The summation in (40) must be performed for three vertices of the T_1, T_2, T_3 in the counterclockwise sense. The components of the unit outer normal $\boldsymbol{n}_Q$ are denoted in the formula (40) as A, B, C

$$\boldsymbol{n}_Q \equiv (A, B, C), \qquad \sqrt{A^2 + B^2 + C^2} = 1, \tag{41}$$

because the components of $\boldsymbol{n}_Q$ are direction cosines of unit vector. There is a simple way to calculate components of $\boldsymbol{n}_Q$ using the vector product of contour vectors of the triangles in Fig. 2, namely:

$$\boldsymbol{n}_Q = \boldsymbol{T}_3\boldsymbol{T}_4 \times \boldsymbol{T}_3\boldsymbol{T}_1 / \left|\boldsymbol{T}_3\boldsymbol{T}_4 \times \boldsymbol{T}_3\boldsymbol{T}_1\right| . \tag{42}$$

The triangle ΔF_j is situated in the plane $t(x, y, z)$ with analytical equation:

$$Ax + By + Cz + D = 0, \tag{43}$$

while D can be calculated by using co-ordinates of some vertex, e.g. $T_1 \equiv (x_1, y_1, z_1)$:

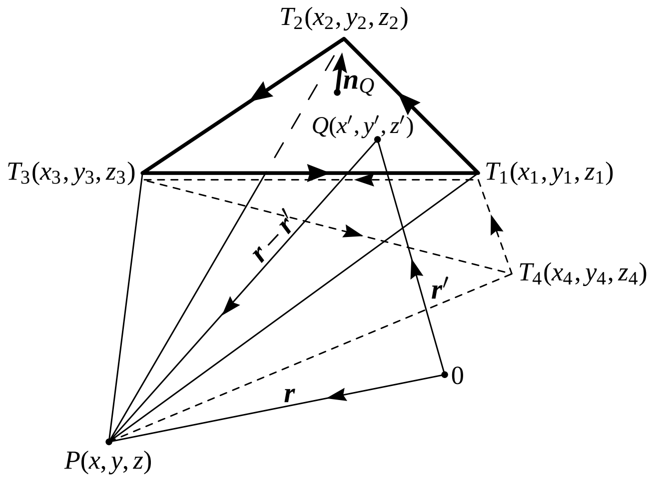


Fig. 2. The parameters for calculation of solid angle for triangular or quadrangle subarea.

$$D = -(Ax_1 + By_1 + Cz_1). \tag{44}$$

Next we calculate the distance of the point $P(x, y, z)$ from the plane $t(x, y, z)$ of triangle:

$$q = |Ax + By + Cz + D|. \tag{45}$$

This distance will be non-zero if the point does not lie in the plane $t(x, y, z)$ and in this case it will hold for the scalar product: $\boldsymbol{n}_Q \cdot (\boldsymbol{r} - \boldsymbol{r}') \neq 0$. If the point P is situated in the plane of the triangle there is $\boldsymbol{n}_Q \cdot (\boldsymbol{r} - \boldsymbol{r}') = 0$ and the solid angle $\Delta\Phi_j$ will be zero as follows from (39). This property must be considered in the program code. In the Ivan's formula (40) there must be used also the following quantities for the neighbouring points T_1, T_2 and $P(x, y, z)$:

$$|\boldsymbol{T}_1\boldsymbol{T}_2| = d_{12} = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2},$$

$$R_1 = \left[(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2\right]^{1/2},$$

$$R_2 = \left[(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2\right]^{1/2}.$$

Then we use the unit vector $\boldsymbol{t}_{12}$ in the direction $\boldsymbol{T}_1\boldsymbol{T}_2$ and the vector $\boldsymbol{PT}_1$ with components:

$$\boldsymbol{t}_{12} \equiv (x_2 - x_1, y_2 - y_1, z_2 - z_1)/d_{12}, \quad \boldsymbol{PT}_1 \equiv (x_1 - x, y_1 - y, z_1 - z).$$

In the next steps the following quantities are calculated:

$$PD_1 = \boldsymbol{PT}_1 \cdot \boldsymbol{t}_{12}, \quad d_2 = PD_1 + d_{12}, \quad \boldsymbol{e}_{12} = \boldsymbol{t}_{12} \times \boldsymbol{n}_Q, \quad w_{12} = \boldsymbol{PT}_1 \cdot \boldsymbol{e}_{12}.$$

This procedure is repeated in the cycle for vertices T_2 and T_3, so we obtain required values for the formula (40). It must be stressed that this algorithm, when applied to the whole closed boundary S (with piecewise continuous normal $\boldsymbol{n}_Q$), must give with high precision, better than 10^{-3}, the well-known fundamental values of the Gauss integral:

$$\int_S \frac{\partial}{\partial n_Q} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \,\mathrm{d}\, s' = \int_S \frac{\boldsymbol{n}' \cdot (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \,\mathrm{d}\, s' = \begin{cases} 0, & P(\boldsymbol{r}) \in \mathrm{Ext}(S) \\ -2\pi, & P(\boldsymbol{r}) \in S \\ -4\pi, & P(\boldsymbol{r}) \in \mathrm{Int}(S). \end{cases} \tag{46}$$

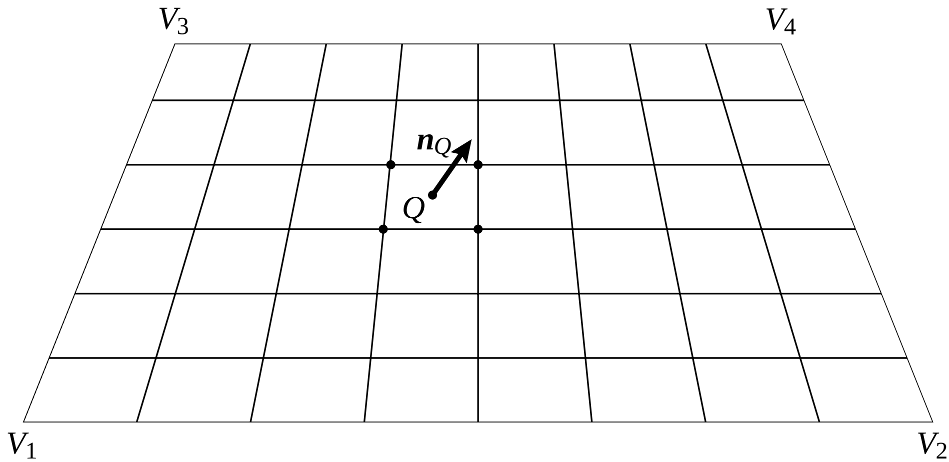


Fig. 3. Scheme of the quadrilateral face of the prismoid and its subdivision into subareas. Note that opposite sides are divided into equal number of intervals, e.g. sides V_1, V_2 and V_3, V_4 have 8 segments.

For the calculation of integral (39) by means of (40) we successfully adopted our original subroutine SLAGIV3 and tested it for the prismoids, like that shown in Fig. 1. We have found that the subdivision of the sloped planar faces of the prismoid into a set of triangle subareas is rather awkward and leads to a large number of subareas. So we decided to improve the Ivan's algorithm into quadrilateral subareas ΔS_j, with four vertices T_1, T_2, T_3, T_4, while normal $\boldsymbol{n}_Q$ is constant for the whole face of prismoid. In this manner we decrease the number of subareas to one half in comparison with triangle case ΔF_j. The algorithm of the subdivision for each of 6 faces into quadrilateral subareas is much simpler and faster. The scheme of subdivision of some quadrilateral face of the prismoid is shown in Fig. 3. The subroutine SLAGIV4 gives values of the Gauss integral (30) i.e. $(-2\pi, -4\pi, 0)$ with accuracy of at least 4 decimal digits. This subroutine was also used for numerical calculations of the forward geoelectrical problem presented in *Hvoždara (2007)*. Let us note that the demands on the computing time and memory were greater than for the similar problem with rectangular faces, because of the more complicated algorithm for the calculation of the solid angle $\Delta\Phi_j$. In the numerical calculations there is necessary to store x, y, z coordinates of vertices for each subarea, as well as coordinates of its centre, which increases demands on computer time and memory. Also we note that the components of $\boldsymbol{n}_Q$ are the same for every subarea ΔF_j of the quadrangle face of the prismoid.

5. Numerical calculations and discussion

The numerical calculations were performed in a similar way as in *Hvoždara (1995, 2007)* regarding that the Green's function $G_1(P, Q)$ is given by the infinite series (34). Nevertheless, the principal terms are again R^{-1}, R_+^{-1} and $R_h^{-1} = [r^2 + (2h - z - z')^2]^{-1/2}$. The special cases when the perturbing body Ω_T touches the bottom and/or upper planar boundary of the layer '1' must be treated similarly as in *Hvoždara and Valkovič (1999)*. The B.I.E. (12) can be solved by the collocation method. It means that the surface S of the perturbing body is discretized into M subareas ΔS_j whose centres we denote as P_m or Q_j. It is also assumed that each subarea is small enough to put $f(Q) = f(Q_j) = \text{const}$ on it. So we introduce the constant approximation of an unknown function $f(Q)$ on ΔS_j. Taking the number M sufficiently large, we can express the B.I.E. (38) in its discretized form:

$$f(P_m) = 2\gamma[V_1(P_m) - v_0] + \sum_{j=1}^{M} f(Q_j)W(P_m, Q_j), \quad m = 1, 2, \ldots, M. \tag{47}$$

Here $\gamma = \beta$ if the body does not touch at the point P_m the planar boundary of the surrounding layer and attains slightly changed values as given in *Hvoždara and Valkovič (1999)* if the prismoid is in contact with planar boundary $z = 0$ or $z = h$. The weighting coefficients $W(P_m, Q_j)$ are given by the formula:

$$W(P_m, Q_j) = \frac{\gamma}{2\pi} \int\!\!\!\int_{\Delta S_j} \frac{\partial}{\partial n_Q} G_1(P_m, Q)\, \mathrm{d}\, S_Q\, . \tag{48}$$

The integration in the principal value sense was explained in comment to the formula (36). It follows that $W(P_m, Q_j)$ cannot be infinite even if $P_m \equiv Q_m$.

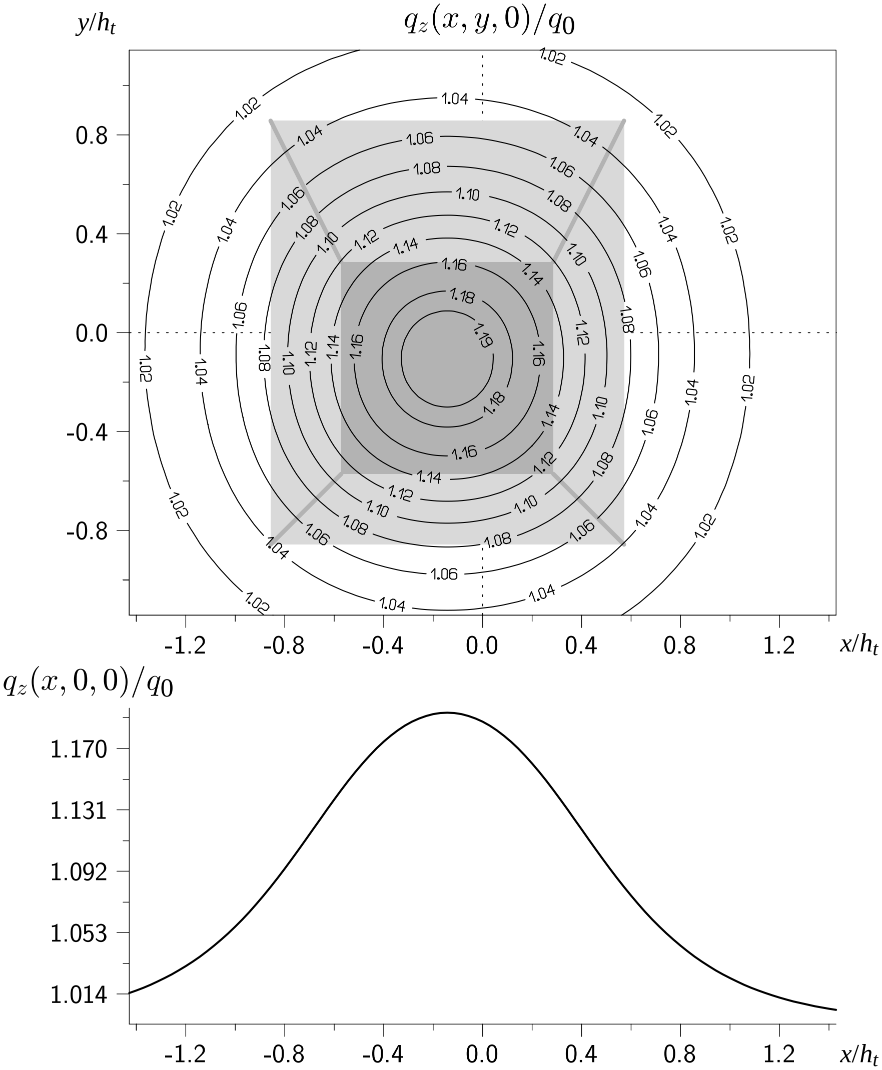
In fact, the formula (47) is the system of M linear equations for the unknown values $f(Q_j)$. This system can be expressed as follows:

$$\sum_{j=1}^{M} [\delta_{mj} - W(P_m, Q_j)] \, f(Q_j) = 2\gamma \, [V_1(P_m) - v_0] \ , \qquad m = 1, 2, ..., M \ , \qquad (49)$$

where δ_{mj} is the Kronecker symbol. This system of equations can be solved using the known methods of linear algebra. Once the system (49) is solved, we can calculate the temperature field and other geothermal characteristics, namely the heat flow density q_z or its anomaly q_z^*.

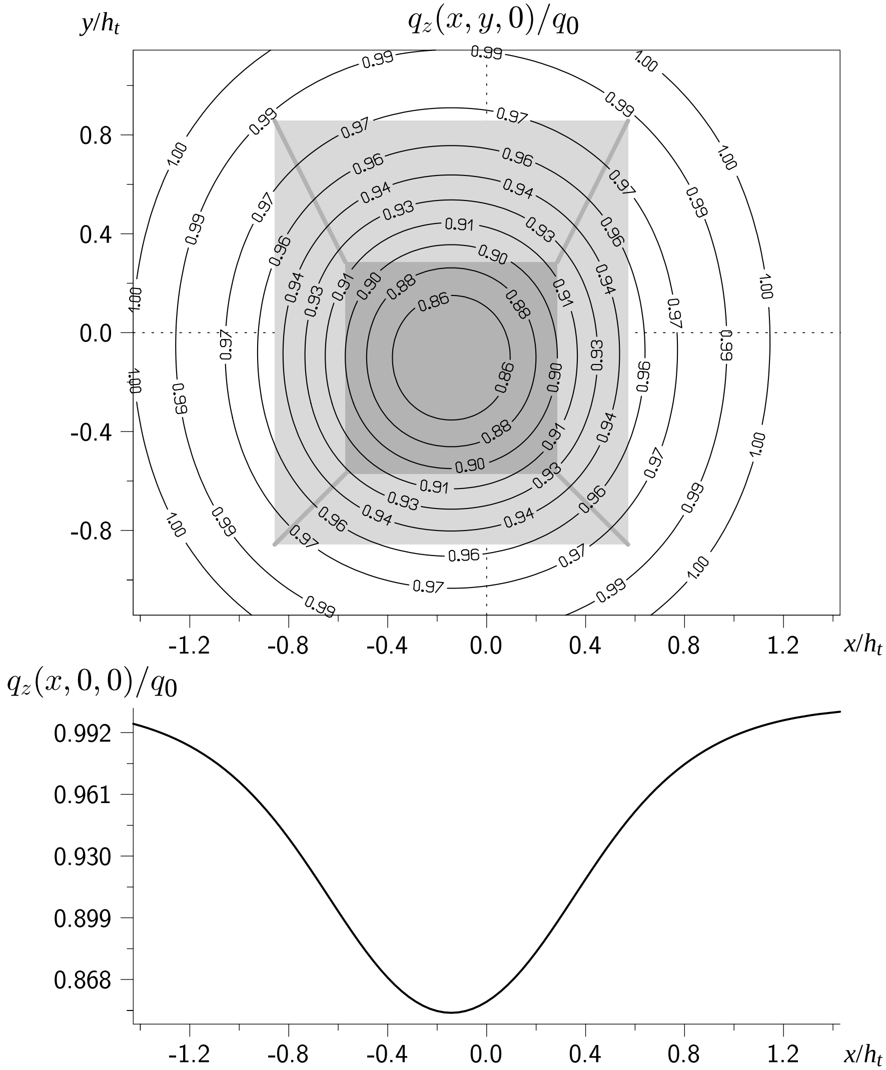
We checked out this algorithm for a prismoid with rectangular bottom and top face while side faces are quadrilaterals. The upper face in the form of rectangle is at the depth z_1, the bottom rectangle is at the depth $z_2 = h$, so the prismoid is in contact with the bottom substratum. The central depth plane of the prismoid is $h_T = (z_1 + z_2)/2$ and we must keep conditions: $z_1 > 0$, $z_2 \leq h$. This block is situated in the first layer with thermal conductivity λ_1, its thickness being h. The thermal conductivity of the prismoid is set at $\lambda_T = \lambda_2$, as some model of penetration of the bottom medium into the superficial layer. In the case $\lambda_T/\lambda_1 > 1$ the prismoid represents a high conductive dyke of the substratum into the layer, and in the case $\lambda_T/\lambda_1 < 1$ the prismoid (dyke) and substratum are of lower thermal conductivity. In our numerical calculations we put $\lambda_1 = 1.0\,\mathrm{W/(K\,m)}$ and $\lambda_2 = \lambda_T = 2.0$ or $0.5\,\mathrm{W/(K\,m)}$.

The subdivision of each face was performed by introducing numbers of division (> 5) for edges of each pair of opposite sides of the trapezoid, which is a general form of some face of the prismoid as shown in Fig. 3. The x, y, z coordinates of vertices for each subarea in the form of quadrangle are stored, since they are used as vertices T_1, T_2, T_3, T_4 for repeated calls of calculation of the solid angle of view by means of subroutine SLAGIV4. The direction cosines of the unit normal $\boldsymbol{n}_Q$ remain constant for each trapezoidal planar face of the prismoid. Let us note that for solving the system of linear equations (49) for each of the central points P_m the weighting coefficients $W(P_m, Q_j)$ must be calculated for all sets of point Q_j, while in Green's function we must treat by using SLAGIV4 at least the contributions from the terms with R^{-1}, R_+^{-1}, and for P_m from the bottom face also from R_h^{-1}. If we choose the subdivision of each trapezoidal face into 64 quadrangle subareas, we obtain $6 \times 64 = 384 = M$ surface elements ΔS_j, which contribute into the summation approximation of the boundary integrals. The heat flow density at the plane $z = 0$ was calculated by means of temperature



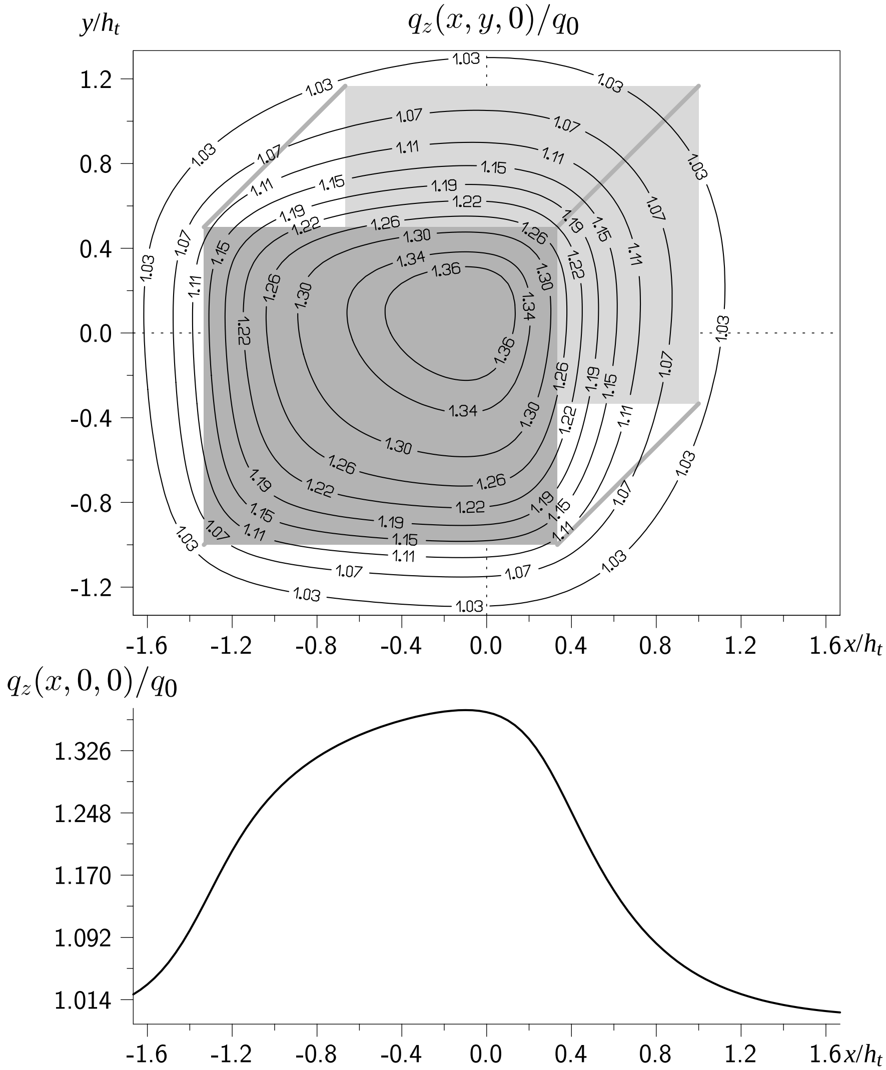
z_1, xl, xr, yl, yr = 1.00 -1.00 .50 -1.00 .50 m
z_2, xl, xr, yl, yr = 2.50 -1.50 1.00 -1.50 1.50 m
h, h_t = 2.50 1.75 m
λ_1 = 1.0, λ_2 = 2.0, λ_T = 2.00 W/(K m)

Fig. 4a. Isolines and profile curve of q_z/q_0 for model prismoid with parameters given in bottom box table and $\lambda_T/\lambda_1 > 1$.



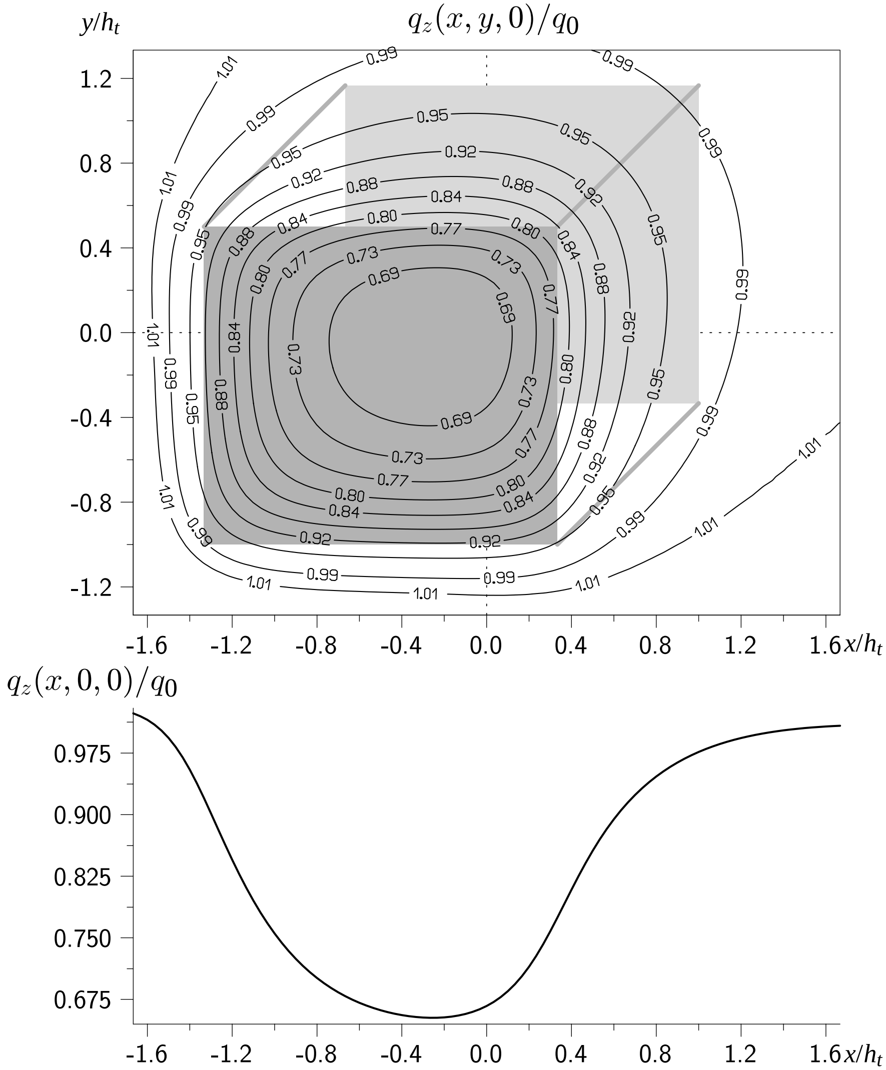
z_1, xl, xr, yl, yr = 1.00 -1.00 .50 -1.00 .50 m
z_2, xl, xr, yl, yr = 2.50 -1.50 1.00 -1.50 1.50 m
h, h_t = 2.50 1.75 m
$\lambda_1 = 1.0$, $\lambda_2 = .5$, $\lambda_T = .50$ W/(K m)

Fig. 4b. Isolines and profile curve of q_z/q_0 for model prismoid with parameters given in bottom box table and $\lambda_T/\lambda_1 < 1$.



z_1, xl, xr, yl, yr = .50 -2.00 .50 -1.50 .75 m
z_2, xl, xr, yl, yr = 2.50 -1.00 1.50 -.50 1.75 m
h, h_t = 2.50 1.50 m
λ_1 = 1.0, λ_2 = 2.0, λ_T = 2.00 W/(K m)

Fig. 5a. Isolines and profile curve of q_z/q_0 for more irregular prismoid as in Figs. 4a,b. Its parameters are given in bottom box table, there is $\lambda_T/\lambda_1 > 1$.



z_1, xl, xr, yl, yr = .50 -2.00 .50 -1.50 .75 m
z_2, xl, xr, yl, yr = 2.50 -1.00 1.50 -.50 1.75 m
h, h_t = 2.50 1.50 m
λ_1 = 1.0, λ_2 = .5, λ_T = .50 W/(K m)

Fig. 5b. Isolines and profile curve of q_z/q_0 for more irregular prismoid as in Figs. 4a,b. Its parameters are given in bottom box table, there is $\lambda_T/\lambda_1 < 1$.

$U_1(x, y, \Delta z)$ where $\Delta z = z_1/10$ is small increment of the depth (z_1 is the depth of upper face of the prismoid). Then we have

$$q_z(x, y, 0) = \lambda_1 [U_1(x, y, \Delta z) - U_0]/\Delta z, \tag{50}$$

while $U_0 \equiv 0^\circ\,\mathrm{K}$ in accordance with the boundary condition (8). The anomalous heat flow q_z^* is simply related to the total q_z by relation $q_z^* = q_z(x, y, 0) - q_0$ so we do not present its graphs. The values of $q_z(x, y, 0)$ were normalized to the constant q_0, the unperturbed heat flow density. For better resolution we present the isoline graphs of q_z/q_0 and also the profile curves along the profile $y = 0$.

The results of our numerical calculations are shown in Figs. 4a,b for highly conductive prismoid (Fig. 4a) and for low conductive prismoid (Fig. 4b). In these figures we show the upper rectangle face of the body by more intense gray and bottom rectangle face by pale gray. The parameters of the model prismoid are given in the box tables in each figure, namely: for the upper rectangle face of the prismoid at the depth z_1, xl, xr, yl, yr are the x, y coordinates (left, right) of the corners; similar by z_2, xl, xr, yl, yr concern the bottom rectangle of the prismoid at the depth $z_2 = h$. The 3-D prismoid in calculations for Figs. 4a,b is similar to the body shown in Fig. 1, like cutted pyramide, with the upper rectangle in the depth $z_1 = 1\,\mathrm{m}$ and the bottom are in the depth $z_2 = 2.5\,\mathrm{m}$. As we expected we obtained the increased (about 20%) values of q_z above the prismoid for $\lambda_T > \lambda_1$ and for $\lambda_T < \lambda_1$ these values are decreased (about 15%). Figs. 5a,b present isoline maps and profile curves for the more irregular prismoid with shallower depth z_1 of the upper face. Comparing to Figs. 4a,b we can find more intensive disturbation of q_z (about 40%) and irregularity of isolines. In this manner we have proved applicability of the BIE method also for prismoids of more general shape than rectangular prism.

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