The connection of an old geodetic datum with a new one using Least Squares Collocation: The Greek case

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Abstract: In the present paper we are dealing with the problem of the connection of two local geodetic datums in which the classical transformation procedures fail to give sufficient solution. Thereafter, we additionally implemented Least Squares Collocation in order to model the remaining residuals of the initial transformation. We tested this methodology in an area located in Northern Greece and we found significant improvement of the transformation results. The implementation of Least Squares Collocation refines the statistical behaviour of the inconsistencies regarding their extreme values (minimum and maximum) from almost 70 to 23 cm, while the standard deviation is reduced from the level of 27 to 10 cm.

Key words: Geodetic Reference System, Least Squares Collocation, Datum, Coordinate Transformation

1. Introduction

The transformation between different geodetic reference systems (or datums following the classical terminology) was and remains a hot spot project for the daily practice of surveyors. A great number of papers, studies and reports deal with this problem (e.g. *Collier et al., 1998; Cai, 2000*). The rapid development of GNSS technology has brought an extra requirement for the connection of the classical local geodetic datums with the more modern global Terrestrial Reference Frames (TRFs).

In the majority of cases, the 2-D or 3-D similarity transformation (e.g. *Strang and Borre, 1997*) is used in order to connect a local geodetic reference frame to a TRF. Nevertheless, it is rather possible that the similarity

doi: 10.1515/congeo-2017-0003

transformation cannot absorb all the blunders, systematic effects and inconsistencies, thus leading to high residuals. Thereafter, some additional mathematical tools were developed to model the remaining transformation residuals such as: Least Squares Collocation (LSC, e.g. *Collier et al., 1998; You and Hwang, 2006; Li et al., 2012*), Finite Elements Model (FEM, e.g. *Kohli and Jenni, 2008*) or other models (e.g. *Dewhurst, 1990*).

In the present paper, we discuss a transformation procedure between two local geodetic reference frame implementations for a region in Greece; the official geodetic datum, called the Hellenic Geodetic Reference System of 1987 (HGRS1987) and the "old Bessel". For this purpose, we use an initial transformation step and a prediction step for the remaining residuals. For validation purposes, we apply the suggested algorithm to a map sheet of "old Bessel" of 1:1000 scale located in the northern part of the country in order to align the old datum to the HGRS1987.

2. Local geodetic datums in Greece

As far as Greece is concerned there are three geodetic reference systems for civil use. Namely:

(1) The newest one is the Hellenic Terrestrial Reference System of 2007 (HTRS07, *Katsambalos et al., 2010*). It is a densification of the European Terrestrial System of 1989 (ETRS89, *Boucher and Altamimi, 2011*). The coordinates of the CORS GPS network, called the Hellenic Positioning System (HEPOS) refer to HTRS07. For its practical implementation, the ellipsoid of GRS80 and the Universal Transverse Mercator cartographic projection (one zone) are used.

(2) The HGRS1987 (*HEMCO, 1987; Veis, 1996*). It was developed in the late '80s, assimilating classical and satellite observations (SLR and GPS). It is connected to the HTRS07 with an accuracy of 8.3 cm, nationwide (*Katsambalos et al., 2010*). It uses the ellpsoid of GRS80 and the Transverse Mercator (one zone).

(3) The old Greek datum (GR-Datum, e.g. *Takos, 1989*). There are two versions of the GR-Datum:

- The so called “new Bessel”. It was realized in the mid-80s after applying new adjustment strategies (*Takos, 1989*), using only classical observations. The ellipsoid of Bessel of 1841 was selected, while its equidistant projection Hatt (*Mugnier, 2002*) was used. The Hatt is a kind of equidistant projection, which inevitably leads to a great number of map sheets. The country was divided into ellipsoidal trapezoids of $30' \times 30'$ (1:100000 scale). Totally, 137 map sheets were produced. Each map sheet holds a different coordinate system, causing a lot of confusion for the surveyors. The “new Bessel” is connected to HGRS1987 through 2nd degree polynomials referred to each sheet (*HEMCO, 1995*).

- The so-called “old Bessel”. It was realized before the Second World War and it carries significant inconsistencies and systematic effects. The great majority of the rural areas in the northern part of the country referred to the “old Bessel”. It uses the Bessel 1841 ellipsoid and it was realized by the division of the northern part of the country into ellipsoidal trapezoids of $6' \times 6'$. The derived mapping infrastructure of “old Bessel”, were surveying layouts of 1:1000 and 1:5000 scale, respectively. Each of the $6' \times 6'$ trapezoids has its own coordinate system. In addition, the associated classical observations were collected during mid ’30s. Till now, there is no officially accepted transformation procedure to connect the “old Bessel” to any of the existing datums, not even with the “new Bessel”. When a surveyor confronts the transformation problem between the “old Bessel” and the HGRS1987 he/she often uses in-situ techniques, focusing only on a limited area of interest (e.g. one or two city-blocks).

3. Mathematical models

The straightforward choice for the connection of two datums is to initially apply some well-known tools, such as the 2D similarity or polynomial transformations. By this step, one can model the major part of their systematic differences. Let us briefly describe these two tools.

3.1. The similarity transformation

Let as assume that we have two sets of projection coordinates. X, Y referring to the new system and x, y referring to the old system, respectively. The 2D similarity transformation model yields (point-wise):

$$\begin{aligned} X_i &= \mu \cos\theta x_i + \mu \sin\theta y_i + t_x + e_{x_i} \,, \\ Y_i &= -\mu \sin\theta x_i + \mu \cos\theta y_i + t_y + e_{y_i} \,, \end{aligned} \tag{1}$$

where μ and θ are the scale and the rotational angle which connect these two systems, t_x, t_y the translations between the two systems with respect to the x and y axes, respectively. Finally, the terms e_{x_i}, e_{y_i} refer to the inherited coordinate errors. After a classical least squares adjustment (e.g. *Koch, 1987*), the optimal parameters (scale, angle and translations) are estimated. Its advantage is the fact that the object's shape between the two systems is preserved.

3.2. The polynomials of second degree

Alternatively, one can apply the 2nd degree polynomials (e.g. *Dermanis and Fotiou, 1992; HEMCO, 1995*) in order to connect the new and the old systems, respectively (point-wise):

$$\begin{aligned} X_i &= a_0 + a_1 x_i^2 + a_2 y_i^2 + a_3 x_i y_i + a_4 x_i + a_5 y_i + e_{x_i} \,, \\ Y_i &= a_6 + a_7 x_i^2 + a_8 y_i^2 + a_9 x_i y_i + a_{10} x_i + a_{11} y_i + e_{y_i} \,, \end{aligned} \tag{2}$$

where $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}$ are the 12 parameters which are estimated through the least squares adjustment. The use of the polynomials has the advantage of adding 8 more parameters than the similarity transformation for the mapping of the systematic effects. On the other hand, the use of the polynomials does not preserve the shape for the objects when it is implemented.

3.3. The LSC implementation

The use of the LSC is widely used in Geodesy. It can serve either for the explicit quantity estimation, for example, geoid (*Moritz, 1980*) or for prediction procedure. The advantage of the LSC is its property of the minimum square error of the prediction (*Koch, 1987*). The mathematical formulation for the LSC implementation is (*Moritz, 1972; Tscherning, 1976*):

$$\hat{\mathbf{s}}' = \mathbf{C}_{\mathbf{s}'\mathbf{s}} \left(\mathbf{C}_{\mathbf{ss}} + \mathbf{D}\right)^{-1} \mathbf{s}\,, \tag{3}$$

where $\hat{\mathbf{s}}'$ the vector of the predicted values, $\mathbf{s}$ the vector of the observations, $\mathbf{C}_{\mathbf{s}'\mathbf{s}}$ the cross-covariance matrix between the predicted values and the observations, $\mathbf{C}_{\mathbf{ss}}$ the covariance matrix of the observations and $\mathbf{D}$ the matrix of the observations noise. In our case, the observations are the reduced residuals with respect to their mean average (we subtract the residual's mean average from each residual component). The residuals are derived after the application of the initial transformation step (similarity transformation or 2nd degree polynomials, respectively). We should underline that Eq. (3) refers to the signal prediction (predicted values). However, in our case, we are dealing with residuals, which by their very nature cannot be separated into signal and noise, as it is expressed through Eq. (3). The application of the least squares adjustment detrends the systematic inconsistencies between the two systems. The residuals cannot be considered as classical observations, since they are the estimated errors of the least squares adjustment. We should additionally note that for this particular application, the noise of the observations is difficult to accurately define. It is rather possible that the older observations are severely affected from various systematic errors. On the other hand, the classical least squares collocation (e.g. applied for the geoid computation) refers to observed quantities like the gravity measurements which comprise the pure signal plus the associated noise.

For the non common points, the predicted coordinates will be estimated according to the following general formula (pointwise):

$$\mathbf{X}_i' = f\left(\mathbf{x}_i', \mathbf{r}_i\right) + \hat{\mathbf{s}}_{\mathbf{x}_i}' + m\,, \tag{4}$$

where $\mathbf{X}_i'$ the coordinates with respect to the new reference frame, $\mathbf{x}_i'$ the coordinates with respect to the old reference system, $\mathbf{r}_i$ the estimated parameters of the initial transformation step, f is the symbol of the function which is related to the first transformation step, $\hat{\mathbf{s}}_{\mathbf{x}_i}'$ the predicted values after the LSC application and m the mean average of the estimated residuals from the initial transformation step.

As a general rule, one should initially compute the so called empirical variance and covariance functions, as follows (*You and Hwang, 2006*):

$$C_0 = \frac{1}{n_k}\sum_{l=j}^{n_k} s_l^2 \tag{5a}$$

$$C_{ss}\left(d_k\right) = \frac{1}{n_k}\sum_{i<j}^{n_k} s_i s_j \tag{5b}$$

where d_k is a predefined distance interval which comprises a number of observation points which lay inside this interval and n_k the number of distances in each interval. Finally, the empirically derived function is fitted to an analytical mathematical expression (e.g. Gaussian, exponential or other numerous functions) using the least squares.

4. Numerical application

The aforementioned methodology is applied to a map sheet located in Northern Greece, called Perithorion (a village in the Prefecture of Drama). Figures 1 and 2 illustrate the village's position and the map sheet of Perithorion

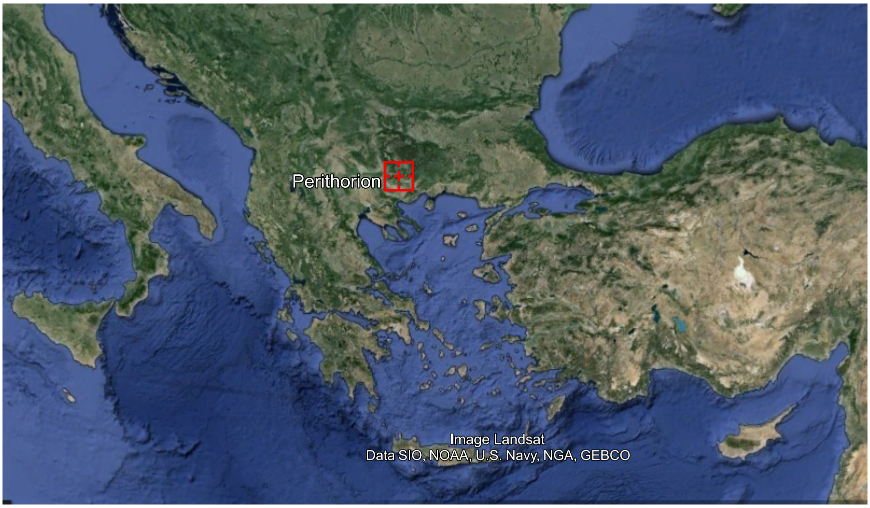

Fig. 1. The village of Perithorion (Google Earth).

Fig. 2. The associated map sheet of Perithorion with respect to the "old Bessel" (scale 1:1000, credits to the Ministry of Agriculture and Foods).

with respect to the "old Bessel" datum, respectively.

The map sheet (scale 1:1000) was digitized from the state agency (Ministry of Agricultural Development and Foods) with respect to its own coordinate system. The map resolution is at the level of 20 cm. This is defined from the fact that the associated eye resolution is approximately 0.2 mm (e.g. *Davidson, 1993*). Applying a simple rule of thumb, we can imply that the points which lay at a map of scale 1:1000 can be identified with an accuracy of 20 cm, according to the following formula:

$$\sigma_{map} = p \times 0.02 \text{ cm}\,, \tag{6}$$

where σ_{map} the accuracy of the derived cartographic coordinates and p the denominator of the scale (in our case 1000). We identified sixty points on the map which nowadays exist (laying at building edges, roads, fences). These points were remeasured via RTK in the spring of 2015, referring to the HGRS1987. Ten of these points were not included in the transformation procedure; they were used as cross validation points for the methodology's assessment.

Initially, we implemented two different options for the first step: The classical 2-D similarity transformation and the 2nd degree polynomials, described in the previous section. Tables 1 and 2 present the residuals using the 2D similarity and 2nd degree polynomials, respectively, while Figure 3 depicts their associated residuals of the Helmert and polynomial transformations.

The residual statistics reveal that there are biases which contaminate the results of both transformations. The minimum and the maximum values for both cases exceed 50 cm in an absolute sense, while the standard deviations are always larger than 20 cm. In addition, the application of

Table 1. The residuals statistics after applying the 2D similarity transformation (from "old Bessel" to HGRS1987) for each coordinate component. Values are in cm.

	δX	δY
min	−78.9	−77.9
max	75.8	70.4
mean	0.3	0.2
std	27.5	31.2

Table 2. The residuals statistics after applying the 2^{nd} degree polynomials (from "old Bessel" to HGRS1987) for each coordinate component. Values are in cm.

	δX	δY
min	−67.1	−68.5
max	68.5	68.7
mean	0.2	0.2
std	26.8	27.4

the 2^{nd} degree polynomials does not offer significantly better performance. Obviously, there are undetectable systematic effects which contaminate the transformation.

In the second step we proceeded with the LSC implementation. The set of the 2D residuals were reduced with respect to their mean average. We used the residuals of the second degree polynomials (which show slightly better statistical behaviour) and we fit the following cardinal sine function (or sinc or sampling function, see e.g. *Brown and Churchill, 1993*) to the empirical one (d is the distance expressed in km):

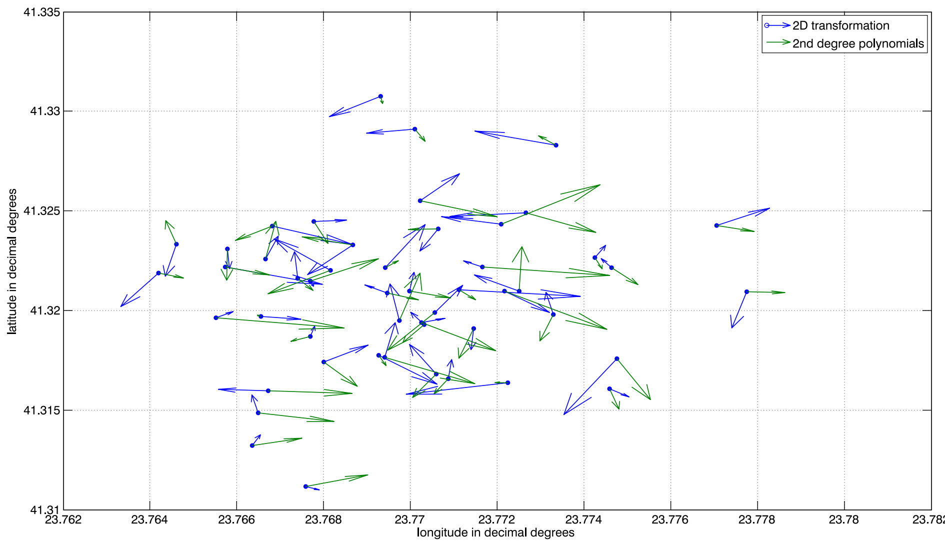

Fig. 3. The residuals of the 2D similarity and the polynomial transformations, respectively.

$$C_{ss}(d) = C_0 \left(\frac{k}{d}\right) \sin\left(\frac{d}{k}\right) . \tag{7}$$

The fitted parameters were the covariance C_0 and the factor k. For our case, the adjusted parameters are:

$\hat{C}_0 = 940.5642\ \mathrm{cm}^2$
$\hat{k} = 0.0395\ \mathrm{km}$

The estimated covariance function was used for the error prediction for both the coordinates' components (X and Y). We should underline that the sinc function is widely used in the signal processing and the Fourier transformations. In our case, it optimally represents the errors' inhomogeneity and roughness, respectively.

Figure 4 depicts the empirical and the fitted covariance functions. We used the cardinal sine function because the simpler Gaussian or the exponential ones are not suitable for this specific case. The use of the Gaussian or exponential function cannot sufficiently fit the data, due to the extremely rough character of the errors. We also tested the Hirvonen function (*Hirvonen, 1962*) but it still presented major deficiencies regarding the

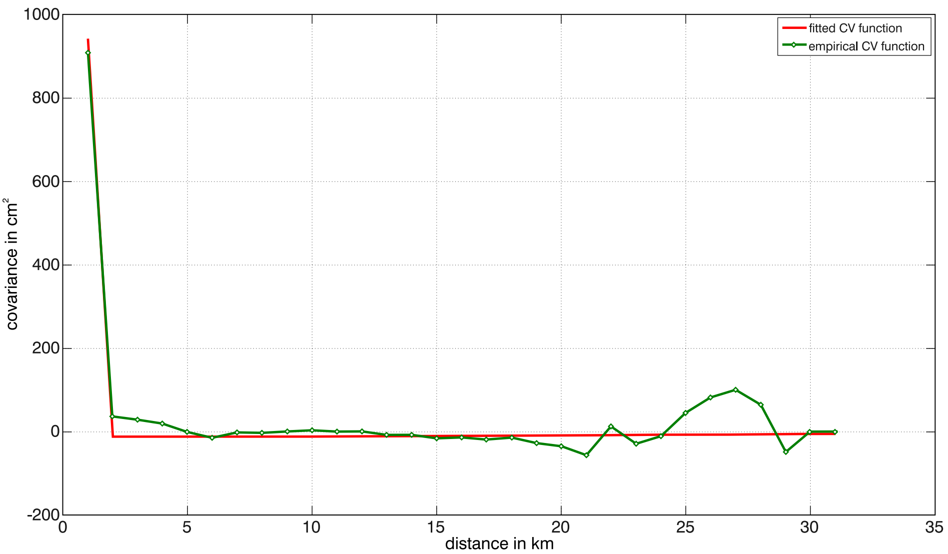

Fig. 4. The empirical and the fitted covariance functions.

fitting procedure.

In order to evaluate the results, we compared the results of the suggested methodology with the coordinates we obtained via RTK at the ten cross validation points. We implemented the Equation 3 (ibid.) without involving the matrix **D** for the observational noise (please see our discussion in section 3.3).

Table 3 shows the discrepancies between the derived coordinates from the suggested methodology and the estimated coordinates with respect to the HGRS1987 via RTK at the ten cross validation points.

Table 3. The discrepancies of the derived coordinates between the suggested methodology and the measured HGRS1987 coordinates at the 10 cross validation points. Values are in cm.

	δX	δY
min	−23.0	−20.7
max	19.1	17.7
mean	1.0	0.9
std	10.3	9.9

From Table 3 we can imply that the methodology improves significantly the results of the transformation between the two datums. The standard deviation of the differences is approximately 10 cm, while the extreme discrepancies (minimum and maximum) are below the level of 25 cm (in an absolute sense). This practically means that we have almost 3 times improvement regarding the standard deviation and 3 times better behaviour of the extreme differences, respectively. Moreover, the statistical performance for both the coordinate components is quite similar. The mean averages of the discrepancies are found at the level of 1 cm. We should also underline the importance of the observation noise matrix.

5. Conclusions and further considerations

We used the LSC approach in order to model the remaining residuals (after an initial similarity or a polynomial transformation) in the case of an old and problematic datum in Greece. We found that after the similarity and the polynomial transformation, significant biases and discrepancies exist. We applied the LSC approach in order to model the remaining residuals. The use of LSC with observation noise improves significantly the connection quality between the "old Bessel" geodetic datum and the HGRS1987. This is reflected in the standard deviations of the inconsistencies: From 26.8 cm (before the LSC application) then reduced to 10.3 cm (after). The extreme discrepancies also present significant refinement: from 68.7 cm (before the LSC application) to 23 cm (after), in an absolute sense.

We recommend that the suggested methodology could be applied in each map sheet of the "old Bessel". For an effective implementation of the approach, all map sheets of the "old Bessel" should be digitized and a series of well distributed points on the map should be remeasured by RTK, with respect to the HGRS1987. The discussed methodology demands an extensive cooperation between the state agencies and professionals in order to stand as an alternative to the severe problem of connecting the rural areas to the official HGRS1987.

Acknowledgements. Dipl. Ing. Nikolaos Demirtzoglou kindly offered the RTK measurements. Dr. Ing. Grigorios Tsinidis helped us with some mathematical computations. Dr. I. Kohut and an anonymous reviewer are kindly acknowledged for the improvement of the initial manuscript.

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