Refinement of the normal Eötvös matrix and its influence on the estimation of the deflections of the vertical

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Abstract: The elements of the Eötvös matrix, usually determined by torsion balance measurements, are useful in many geodetic applications. We present a method for the computation of the elements of the normal Eötvös matrix at a point on the Earth's physical surface, resulting in an improvement in the determination of the deflection of the vertical at intermediate points of a network. In the process, we also present analytical expressions for the computation of the components of the deflection of the vertical. From those expressions and using also a numerical example, we show that the proposed refinement is not completely negligible.

Key words: Eötvös matrix, deflection of the vertical, normal gravity field.

1. Introduction

The Eötvös matrix is the second order derivative of the Earth's gravity potential at a point P. The Earth's gravity potential is expressed in a local Cartesian system (x, y, z). This system is centred at point P (point of measurement), the z-axis is perpendicular to the equipotential surface passing through point P pointing outwards, the x-axis is tangent to the equipotential surface passing through point P pointing North and the y-axis is tangent to the aforementioned equipotential surface pointing East. Using the letter "W" for the Earth's gravity potential, then its second order derivative at point P expressed in this local Cartesian system is equal to:

$$E(P) = \begin{bmatrix} W_{xx} & W_{xy} & W_{xz} \\ W_{yx} & W_{yy} & W_{yz} \\ W_{zx} & W_{zy} & W_{zz} \end{bmatrix}_P . \tag{1.1}$$

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The elements of the Eötvös matrix (except W_{zz}) are determined by torsion balance measurements at point P. The Eötvös matrix is significant because, for example, it plays an important role for the "Geodetic Singularity Problem": if the determinant of the Eötvös matrix at point P is equal to zero, then it is rank deficient and this classifies point P as a singular point. This means that it is not possible to replace (pseudo)differentials of unholonomic coordinate systems, which are related to moving local astronomical frames, with differentials of holonomic coordinate systems (*Livieratos, 1976*). Another application of the Eötvös matrix is the determination of the deflection of the vertical at points on the Earth's physical surface (*Völgyesi, 1993, 1998*). The elements of the Eötvös matrix which are involved are W_{xx}, W_{xy} and W_{yy}. A third application of the Eötvös matrix is the determination of the geoid undulation by an alternative solution for the astrogeodetic levelling (*Völgyesi, 2001*). Finally, the determination of gravity anomaly is possible (for gravimetric determination of the geoid) with the help of the elements W_{xz} and W_{yz}.

In this work we will describe briefly the method for the determination of the deflection of the vertical with torsion balance measurements. The necessary equations of this method include the diagonal elements W_{xx} and W_{yy} of the Eötvös matrix and also the diagonal elements U_{xx} and U_{yy} of the normal Eötvös matrix at point P (the letter "U" is used for the normal gravity potential generated by an equipotential ellipsoid of revolution – the reference ellipsoid). Until now, the values of the elements U_{xx}, and U_{yy} have been determined on the surface of a chosen ellipsoid of revolution. Here we will present a method for their determination on the Earth's physical surface.

2. Determination of deflections of the vertical with torsion balance measurements

There is a set of points $P_1, P_2, P_3, ..., P_{n-2}, P_{n-1}, P_n$ on the Earth's physical surface covering a relative small area (see Fig. 1). At points P_1 and P_n the deflection of the vertical is known. Torsion balance measurements have been performed at all points. For the triangle $P_1P_2P_3$ we have three equations (*Völgyesi, 1993*):

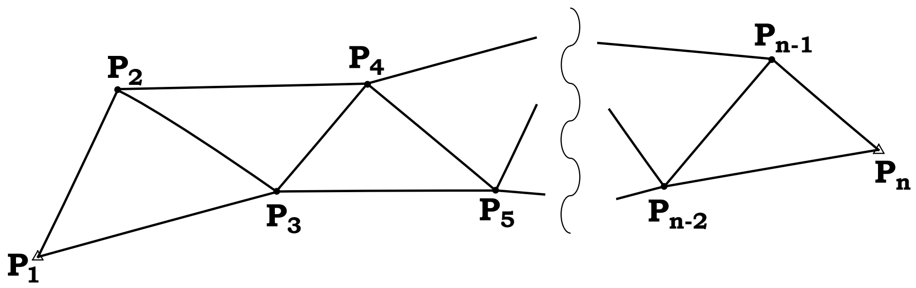

Fig. 1. Torsion balance and gravity measurements network.

$$\begin{aligned}\Delta\xi_{21}\sin\alpha_{12} - \Delta\eta_{21}\cos\alpha_{12} = \\ = \frac{S_{12}}{4g_{12}}\Big[(W_{yy}-U_{yy}+W_{xx}-U_{xx})\Big|_{P_1}\sin 2\alpha_{12} + \\ +(W_{yy}-U_{yy}+W_{xx}-U_{xx})\Big|_{P_2}\sin 2\alpha_{12} + \\ +(W_{xy}-U_{xy})\Big|_{P_1}\cos 2\alpha_{12} + (W_{xy}-U_{xy})\Big|_{P_2}\cos 2\alpha_{12}\Big]\,,\end{aligned} \tag{2.1}$$

$$\begin{aligned}\Delta\xi_{32}\sin\alpha_{23} - \Delta\eta_{32}\cos\alpha_{23} = \\ = \frac{S_{23}}{4g_{23}}\Big[(W_{yy}-U_{yy}+W_{xx}-U_{xx})\Big|_{P_2}\sin 2\alpha_{23} + \\ +(W_{yy}-U_{yy}+W_{xx}-U_{xx})\Big|_{P_3}\sin 2\alpha_{23} + \\ +(W_{xy}-U_{xy})\Big|_{P_2}\cos 2\alpha_{23} + (W_{xy}-U_{xy})\Big|_{P_3}\cos 2\alpha_{23}\Big]\,,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\Delta\xi_{31}\sin\alpha_{13} - \Delta\eta_{31}\cos\alpha_{13} = \\ = \frac{S_{13}}{4g_{13}}\Big[(W_{yy}-U_{yy}+W_{xx}-U_{xx})\Big|_{P_1}\sin 2\alpha_{13} + \\ +(W_{yy}-U_{yy}+W_{xx}-U_{xx})\Big|_{P_3}\sin 2\alpha_{13} + \\ +(W_{xy}-U_{xy})\Big|_{P_1}\cos 2\alpha_{13} + (W_{xy}-U_{xy})\Big|_{P_3}\cos 2\alpha_{13}\Big]\,,\end{aligned} \tag{2.3}$$

where

$$\Delta\xi_{21} = \xi_1 - \xi_2\,, \tag{2.4}$$

$$\Delta\eta_{21} = \eta_1 - \eta_2\,. \tag{2.5}$$

Similar relations hold for $\Delta\xi_{32}$, $\Delta\xi_{31}$, $\Delta\eta_{32}$ and $\Delta\eta_{31}$. Also, α_{12}, α_{23} and α_{13} are the azimuths of P_1P_2, P_2P_3 and P_1P_3, while S_{12}, S_{23} and S_{13} are the lengths of P_1P_2, P_2P_3 and P_1P_3 respectively, and g_{ij} is an average gravity value between P_i and P_j. The local Cartesian system (x, y, z) is centred at point P_1. Additional relations for the triangle $P_1P_2P_3$ are:

$$\Delta\xi_{21} + \Delta\xi_{32} + \Delta\xi_{13} = 0\,, \tag{2.6}$$

$$\Delta\eta_{21} + \Delta\eta_{32} + \Delta\eta_{13} = 0\,. \tag{2.7}$$

For the triangle $P_1P_2P_3$ we have six unknowns which are $\Delta\xi_{21}$, $\Delta\xi_{32}$, $\Delta\xi_{13}$, $\Delta\eta_{21}$, $\Delta\eta_{32}$, and $\Delta\eta_{13}$. Therefore for the $n-2$ triangles (see Fig. 1) we have $4n-6$ unknowns and $4n-7$ equations. We need one more equation to find the unknowns, and we can choose one from the following two equations:

$$\sum_{i=2}^{n} \Delta\xi_{i-1,i} = \xi_n - \xi_1\,, \tag{2.8}$$

or

$$\sum_{i=2}^{n} \Delta\eta_{i-1,i} = \eta_n - \eta_1\,. \tag{2.9}$$

3. Improving the elements of the Eötvös matrix

The second order partial derivatives of the normal potential U_{xx} and U_{yy}, which are involved in the equations (2.1), (2.2), and (2.3) have approximate values, i.e. they are determined on the surface of the chosen reference ellipsoid of revolution and not on the Earth's physical surface. The points of interest $P_1, P_2, P_3, ..., P_{n-2}, P_{n-1}, P_n$ are projected on the ellipsoid along the ellipsoidal normals passing through these points. Let the projection points be $Q_1, Q_2, ..., Q_n$ and (x_{Qi}, y_{Qi}, z_{Qi}) the corresponding local Cartesian systems which are defined on the aforementioned projection points. The value of the components of the normal Eötvös matrix at points $P_1, P_2, ..., P_n$ is given by *Toth et al. (2001)*:

$$U_{yy} - U_{xx} \equiv U_{y_{Qi}y_{Qi}} - U_{x_{Qi}x_{Qi}} = \\ = \gamma\left(\frac{1}{N} - \frac{1}{M}\right) = \gamma(k_2 - k_1) = 10.26\cos^2\phi\,, \tag{3.1}$$

in Eötvös units ($1\,\mathrm{E} = 10^{-9}\mathrm{sec}^{-2}$). The symbol "$\gamma$" stands for the normal gravity on the ellipsoid while M and N are its principal radii of curvature. For the above equation we made the assumption that:

$$U_{yy} - U_{xx} = U_{y_{Qi}y_{Qi}} - U_{x_{Qi}x_{Qi}}\,. \tag{3.1a}$$

In addition, the second assumption for equation (3.1) is:

$$U_{xy} \equiv U_{x_{Qi}y_{Qi}} = 0\,. \tag{3.1b}$$

As we mentioned before, (x_{Qi}, y_{Qi}, z_{Qi}) are the local Cartesian systems at points $Q_1, Q_2, ..., Q_n$. We define a second local Cartesian system at each point on the Earth's physical surface by the following relation:

$$\begin{bmatrix} x_{1i} \\ y_{1i} \\ z_{1i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -h_{P_1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_i & \sin\delta_i \\ 0 & -\sin\delta_i & \cos\delta_i \end{bmatrix} \begin{bmatrix} x_{Q_i} \\ y_{Q_i} \\ z_{Q_i} \end{bmatrix}, \quad i = 1, 2, ..., n\,, \tag{3.2}$$

where (x_{1i}, y_{1i}, z_{1i}), $i = 1, 2, ..., n$ are the related local Cartesian systems at points $P_1, P_2, P_3, ..., P_{n-2}, P_{n-1}, P_n$ and h_{Pi} is the geometric height of point P_i (Fig. 2).

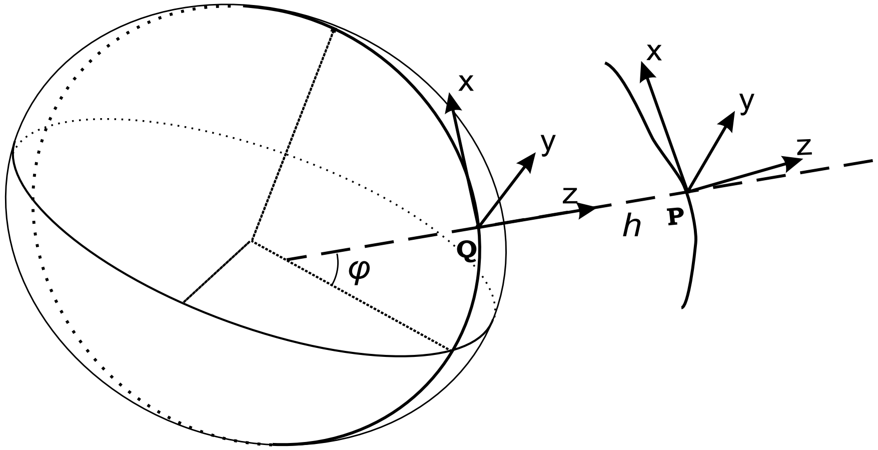

Fig. 2. Coordinate systems at points Q_i and P_i.

From the above relation we also get the following inverse relation (since the rotation matrix is orthogonal):

$$\begin{bmatrix} x_{Q_i} \\ y_{Q_i} \\ z_{Q_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_i & -\sin\delta_i \\ 0 & \sin\delta_i & \cos\delta_i \end{bmatrix} \begin{bmatrix} x_{1i} \\ y_{1i} \\ z_{1i} + h_{P_1} \end{bmatrix}, \quad i = 1, 2, ..., n\,. \tag{3.3}$$

From (3.2) we have:

$$\begin{bmatrix} \frac{\partial x_{Q_i}}{\partial x_{1i}} & \frac{\partial x_{Q_i}}{\partial y_{1i}} & \frac{\partial x_{Q_i}}{\partial z_{1i}} \\ \frac{\partial y_{Q_i}}{\partial x_{1i}} & \frac{\partial y_{Q_i}}{\partial y_{1i}} & \frac{\partial y_{Q_i}}{\partial z_{1i}} \\ \frac{\partial z_{Q_i}}{\partial x_{1i}} & \frac{\partial z_{Q_i}}{\partial y_{1i}} & \frac{\partial z_{Q_i}}{\partial z_{1i}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_i & -\sin\delta_i \\ 0 & \sin\delta_i & \cos\delta_i \end{bmatrix}, \tag{3.4}$$

see *Hofmann-Wellenhof and Moritz (2006).* The angle δ_i reads:

$$\delta_i = -h_{P_i} \frac{f^*}{R} \sin 2\phi_P\,, \tag{3.5}$$

where f^* is the gravitational flattening. The assumption which is made for the determination of the elements of the normal Eötvös matrix is:

$$U_{xx}(P_i) = U_{x_{Qi}x_{Qi}} \tag{3.6}$$

$$U_{yy}(P_i) = U_{y_{Qi}y_{Qi}} \tag{3.6a}$$

We suggest the following refinement for the above elements of the normal Eötvös matrix, see also (*Manoussakis, 2013*):

$$U_{x_{1i}x_{1i}} = U_{x_{Qi}x_{Qi}} \tag{3.6b}$$

$$U_{y_{1i}y_{1i}} = U_{y_{Qi}y_{Qi}} \cos^2\delta_i + U_{y_{Qi}z_{Qi}} \sin 2\delta_i + U_{z_{Qi}z_{Qi}} \sin^2\delta_i \tag{3.7}$$

$$U_{x_{1i}x_{1i}}(P_i) = U_{xx}(P_i) = U_{x_{Qi}x_{Qi}}(Q_i) + U_{x_{Qi}x_{Qi}z_{Qi}}(Q_i) h_{P_i} \tag{3.8}$$

$$\begin{aligned} U_{y_{1i}y_{1i}}(P_i) = U_{yy}(P_i) = {} & [U_{y_{Qi}y_{Qi}}(Q_i) + U_{y_{Qi}y_{Qi}z_{Qi}}(Q_i) h_{P_i}] \cos^2\delta_i + \\ & + [U_{y_{Qi}z_{Qi}}(Q_i) + U_{y_{Qi}z_{Qi}z_{Qi}}(Q_i) h_{P_i}] \sin 2\delta_i + \\ & + [U_{z_{Qi}z_{Qi}}(Q_i) + U_{z_{Qi}z_{Qi}z_{Qi}}(Q_i) h_{P_i}] \sin^2\delta_i \end{aligned} \tag{3.9}$$

where

$$U_{x_{Qi}x_{Qi}z_{Qi}} \equiv \frac{\partial^3 U}{\partial x_{Qi}\partial x_{Qi}\partial z_{Qi}} = \gamma k_1^2 \tag{3.10}$$

$$\frac{\partial \phi}{\partial y_{Q_i}} = \frac{1}{\dfrac{\partial}{\partial \phi}\left(\dfrac{1}{k_1}\right)\left(\dfrac{b^2}{2a^2}-\dfrac{1}{2}\right)\sin 2\phi + \dfrac{1}{k_1}\left(\dfrac{b^2}{a^2}\cos^2\phi + \sin^2\phi\right)} \tag{3.11}$$

$$U_{y_{Qi}y_{Qi}z_{Qi}} \equiv \frac{\partial^3 U}{\partial y_{Qi}\partial y_{Qi}\partial z_{Qi}} = \gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y_{Q_i}^2} \tag{3.12}$$

$$\frac{\partial \gamma}{\partial y_{Q_i}} = \frac{\partial \gamma}{\partial \phi}\frac{\partial \phi}{\partial y_{Q_i}} = \frac{\partial \gamma}{\partial \phi}\frac{1}{\frac{\partial y_{Q_i}}{\partial \phi}} \tag{3.13}$$

$$\frac{\partial^2 \gamma}{\partial y_{Q_i}^2} = \left(\frac{\partial^2 \gamma}{\partial \phi^2} - \frac{\partial \gamma}{\partial y_{Q_i}}\frac{\partial^2 y_{Q_i}}{\partial \phi^2}\right)\frac{1}{\left(\dfrac{\partial^2 y_{Q_i}}{\partial \phi^2}\right)^2} \tag{3.14}$$

From (3.10) and (3.12) we have that:

$$\left(U_{x_{Qi}x_{Qi}x_{Qi}} - U_{y_{Qi}y_{Qi}z_{Qi}}\right)_{Q_i} h_{P_i} = \left[\gamma(k_2^2 - k_1^2) - \frac{\partial^2 \gamma}{\partial y_{Q_i}^2}\right] h_{P_i} = \varepsilon_1 - \varepsilon_2 \tag{3.15}$$

The first term is approximately equal to (see appendix):

$$\varepsilon_1 = -\gamma(k_1^2 - k_2^2)h_{P_i} \cong -2{e'}^2\gamma\cos^2\phi_{P_i}\frac{b^2}{a^4}h_{P_i} \tag{3.16}$$

where e' is the second numerical eccentricity of the reference ellipsoid, γ is the normal gravity value on the surface of the ellipsoid (at point Q_i), a and b the semi axes of the ellipsoid and h_{Pi} the geometric height of point P_i on the Earth's physical surface. We have that:

$$(W_{yy} - W_{xx}) - (U_{yy} - U_{xx}) = (-gk_2^a + gk_1^a) - (\gamma k_2 - \gamma k_1) \tag{3.17}$$

where k_1^a, k_2^a are the values of principal curvatures of the actual equipotential surface of point P_i, and k_1 and k_2 are the principal curvatures of the normal equipotential surface at point Q_i, i.e. the curvatures of the reference ellipsoid itself and γ is the normal gravity value at point Q_i. Making the necessary manipulations (see appendix) we arrive at the following relation:

$$
\begin{aligned}
&(W_{yy} - W_{xx}) - (U_{yy} - U_{xx}) = \\
&\qquad = -\left(\gamma + \delta g(P_i) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_i} h_{P_i}\right)\left(2e'^2 \frac{b^2}{a^4}\cos^2\phi_{P_i}\right) h_{P_i} - \\
&\qquad\quad - \left(\delta g(P_i) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_i} h_{P_i}\right)\frac{b}{a^2} e'^2 \cos^2\phi_{P_i}\,,
\end{aligned}
\tag{3.18}
$$

where $\delta g(P_i)$ is the gravity disturbance at point P_i. The above equation gives an approximation of the difference $W_{yy} - W_{xx} - (U_{yy} - U_{xx})$ which is used in equations (2.1), (2.2) and (2.3). Adding the correction terms, see (3.15) and (3.16), the above relation becomes:

$$
\begin{aligned}
&(W_{yy} - W_{xx}) - (U_{yy} - U_{xx}) = \\
&\qquad = -\left(2\gamma + \delta g(P_i) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_i} h_{P_i}\right)\left(2e'^2 \frac{b^2}{a^4}\cos^2\phi_{P_i}\right) h_{P_i} - \\
&\qquad\quad - \left(\delta g(P_i) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_i} h_{P_i}\right)\frac{b}{a^2} e'^2 \cos^2\phi_{P_i} - \left.\frac{\partial^2 \gamma}{\partial y^2}\right|_{Q_i} h_{P_i}\,.
\end{aligned}
\tag{3.19}
$$

4. Analytical expressions for the deflections of the vertical

For the sake of simplicity, we shall derive analytical expressions for the computation of the deflections of the vertical in a small network of three points only. Let us assume also that the differences $\Delta\xi_{12}$ and $\Delta\eta_{12}$ are known, so we shall present expressions for $\Delta\xi_{31}$, $\Delta\xi_{32}$, $\Delta\eta_{31}$ and $\Delta\eta_{32}$ in two cases: at first using only the classical part of the elements of the normal Eötvös matrix and then including the suggested refinement.

Case I:

The linear system for the determination of the four unknowns $\Delta\xi_{31}$, $\Delta\xi_{32}$, $\Delta\eta_{31}$ and $\Delta\eta_{32}$ is the following:

$$
\begin{aligned}
&\Delta\xi_{31}\sin\alpha_{13} - \Delta\eta_{31}\cos\alpha_{13} = \\
&\qquad = \frac{S_{13}}{4g_{13}}\Big[\left.(W_{yy} - W_{xx} - U_{yy} + U_{xx})\right|_{P_1}\sin 2\alpha_{13} + \\
&\qquad\qquad + \left.(W_{yy} - W_{xx} - U_{yy} + U_{xx})\right|_{P_3}\sin 2\alpha_{13} + \\
&\qquad\qquad + \left.(W_{xy} - U_{xy})\right|_{P_1}\cos 2\alpha_{13} + \left.(W_{xy} - U_{xy})\right|_{P_3}\cos 2\alpha_{13}\Big]\,,
\end{aligned}
\tag{4.1}
$$

$$\Delta\xi_{32}\sin\alpha_{23} - \Delta\eta_{32}\cos\alpha_{23} = \\ = \frac{S_{23}}{4g_{23}}\Big[(W_{yy} - W_{xx} - U_{yy} + U_{xx})\big|_{P_2}\sin 2\alpha_{23} + \\ + (W_{yy} - W_{xx} - U_{yy} + U_{xx})\big|_{P_3}\sin 2\alpha_{23} + \\ + (W_{xy} - U_{xy})\big|_{P_2}\cos 2\alpha_{23} + (W_{xy} - U_{xy})\big|_{P_3}\cos 2\alpha_{23}\Big]\,, \tag{4.2}$$

$$\Delta\xi_{31} + \Delta\xi_{32} = -\Delta\xi^{m}_{21}\,, \tag{4.3}$$

$$\Delta\eta_{31} + \Delta\eta_{32} = -\Delta\eta^{m}_{21}\,, \tag{4.4}$$

where the letter "m" stands for the word "measured". The determinant of the above system is equal to:

$$D = \begin{vmatrix} \sin a_{13} & 0 & -\cos a_{13} & 0 \\ 0 & \sin a_{23} & 0 & -\cos a_{23} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \sin(a_{23} - a_{13})\,. \tag{4.5}$$

Let

$$-\left(\gamma(Q_1) + \delta g(P_1) + \frac{\partial\gamma}{\partial h}\bigg|_{Q_1} h_{P_1}\right)\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_1}\right)h_{P_1} - \\ - \left(\delta g(P_1) + \frac{\partial\gamma}{\partial h}\bigg|_{Q_1} h_{P_1}\right)\frac{b}{a^2}e'^2\cos^2\phi_{P_1} - \frac{\partial^2\gamma}{\partial y^2}\bigg|_{Q_1} h_{P_1} = \\ = -\gamma(Q_1)c_{11} + c_{12}\,, \tag{4.6}$$

$$-\left(\gamma(Q_2) + \delta g(P_2) + \frac{\partial\gamma}{\partial h}\bigg|_{Q_2} h_{P_2}\right)\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_2}\right)h_{P_2} - \\ - \left(\delta g(P_2) + \frac{\partial\gamma}{\partial h}\bigg|_{Q_2} h_{P_2}\right)\frac{b}{a^2}e'^2\cos^2\phi_{P_2} - \frac{\partial^2\gamma}{\partial y^2}\bigg|_{Q_2} h_{P_2} = \\ = -\gamma(Q_2)c_{21} + c_{22}\,, \tag{4.7}$$

$$-\left(\gamma(Q_3)+\delta g(P_3)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_3}h_{P_3}\right)\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_3}\right)h_{P_3}-$$
$$-\left(\delta g(P_3)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_3}h_{P_3}\right)\frac{b}{a^2}e'^2\cos^2\phi_{P_3}-\left.\frac{\partial^2\gamma}{\partial y^2}\right|_{Q_3}h_{P_3}= \tag{4.8}$$
$$=-\gamma(Q_3)c_{31}+c_{32}\,,$$

where

$$c_{11}=\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_1}\right)h_{P_1}\,, \tag{4.8a}$$

$$c_{12}=-\left(\delta g(P_1)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_1}h_{P_1}\right)\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_1}\right)h_{P_1}-$$
$$-\left(\delta g(P_1)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_1}h_{P_1}\right)\frac{b}{a^2}e'^2\cos^2\phi P_1-\left.\frac{\partial^2\gamma}{\partial y^2}\right|_{Q_1}h_{P_1}\,, \tag{4.8b}$$

$$c_{21}=\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_2}\right)h_{P_2}\,, \tag{4.8c}$$

$$c_{22}=-\left(\delta g(P_2)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_2}h_{P_2}\right)\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_2}\right)h_{P_2}-$$
$$-\left(\delta g(P_2)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_2}h_{P_2}\right)\frac{b}{a^2}e'^2\cos^2\phi_{P_2}-\left.\frac{\partial^2\gamma}{\partial y^2}\right|_{Q_2}h_{P_2}\,, \tag{4.8d}$$

$$c_{31}=\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_3}\right)h_{P_3}\,, \tag{4.8e}$$

$$c_{32}=-\left(\delta g(P_3)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_3}h_{P_3}\right)\left(2e'^2\frac{b^2}{a^4}\cos^2\phi_{P_3}\right)h_{P_3}-$$
$$-\left(\delta g(P_3)+\left.\frac{\partial\gamma}{\partial h}\right|_{Q_3}h_{P_3}\right)\frac{b}{a^2}e'^2\cos^2\phi_{P_3}-\left.\frac{\partial^2\gamma}{\partial y^2}\right|_{Q_3}h_{P_3}\,. \tag{4.8f}$$

The solution for $\Delta\xi_{31}$ is found as follows:

$$D_{\Delta\xi 31} = \begin{vmatrix} M_{11} & 0 & -\cos a_{13} & 0 \\ M_{21} & \sin a_{23} & 0 & -\cos a_{23} \\ -\Delta\xi_{21}^{m} & 1 & 0 & 0 \\ -\Delta\eta_{21}^{m} & 0 & 1 & 1 \end{vmatrix}, \tag{4.9}$$

where M_{11} and M_{21} are auxiliary terms:

$$M_{11} = \Big\{[-(\gamma(Q_1)c_{11} + \gamma(Q_3)c_{31}) + c_{12} + c_{32}] \sin 2a_{13} + \\ + [W_{xy}(P_1) + W_{xy}(P_3)] \cos 2a_{13}\Big\} \frac{S_{13}}{4g_{13}},$$

$$M_{21} = \Big\{[-(\gamma(Q_2)c_{21} + \gamma(Q_3)c_{31}) + c_{22} + c_{32}] \sin 2a_{23} + \\ + [W_{xy}(P_2) + W_{xy}(P_3)] \cos 2a_{23}\Big\} \frac{S_{23}}{4g_{23}},$$

and

$$\Delta\xi_{31} = \frac{D_{\Delta\xi 31}}{D}. \tag{4.10}$$

We split the solution in two parts i.e.

$$D_{\Delta\xi 31} = \begin{vmatrix} M'_{11} & 0 & -\cos a_{13} & 0 \\ M'_{21} & \sin a_{23} & 0 & -\cos a_{23} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} + \\ + \begin{vmatrix} M''_{11} & 0 & -\cos a_{13} & 0 \\ M''_{21} & \sin a_{23} & 0 & -\cos a_{23} \\ -\Delta\xi_{21}^{m} & 1 & 0 & 0 \\ -\Delta\eta_{21}^{m} & 0 & 1 & 1 \end{vmatrix} = D_1 + D_{c_1} \tag{4.11}$$

where M'_{11}, M'_{21}, M''_{11} and M''_{21} are auxiliary terms as well:

$$M'_{11} = -\Big\{\gamma(Q_1)c_{11} + \gamma(Q_3)c_{31}\Big\} \frac{S_{13}}{4g_{13}} \sin 2a_{13},$$

$$M'_{21} = -\Big\{\gamma(Q_2)c_{21} + \gamma(Q_3)c_{31}\Big\}\frac{S_{23}}{4g_{23}}\sin 2a_{23}\,,$$

$$M''_{11} = \Big\{[c_{12} + c_{32}]\sin 2a_{13} + [W_{xy}(P_1) + W_{xy}(P_3)]\cos 2a_{13}\Big\}\frac{S_{13}}{4g_{13}}\,,$$

$$M''_{21} = \Big\{[c_{22} + c_{32}]\sin 2a_{23} + [W_{xy}(P_2) + W_{xy}(P_3)]\cos 2a_{23}\Big\}\frac{S_{23}}{4g_{23}}\,.$$

The rationale for the splitting is that, in both cases, the second determinant will remain unchanged and only the first determinant will change. Hence:

$$\Delta\xi_{31} = \frac{1}{\sin(a_{23} - a_{13})}(D_1 + D_{c_1})\,. \tag{4.12}$$

Similar relations hold for the rest of the unknowns i.e.:

$$\Delta\xi_{32} = \frac{1}{\sin(a_{23} - a_{13})}(D_2 + D_{c_2})\,, \tag{4.13}$$

$$\Delta\eta_{31} = \frac{1}{\sin(a_{23} - a_{13})}(D_3 + D_{c_3})\,, \tag{4.14}$$

$$\Delta\eta_{32} = \frac{1}{\sin(a_{23} - a_{13})}(D_4 + D_{c_4})\,. \tag{4.15}$$

Case II:

We now apply the suggested refinement to the elements of the normal Eötvös matrix. The above solution becomes:

$$D^{new}_{\Delta\xi 31} = 2\begin{vmatrix} M'_{11} & 0 & -\cos a_{13} & 0 \\ M'_{21} & \sin a_{23} & 0 & -\cos a_{23} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} + \\ + \begin{vmatrix} M''_{11} & 0 & -\cos a_{13} & 0 \\ M''_{21} & \sin a_{23} & 0 & -\cos a_{23} \\ -\Delta\xi^m_{21} & 1 & 0 & 0 \\ -\Delta\eta^m_{21} & 0 & 1 & 1 \end{vmatrix}, \tag{4.16}$$

and the new solution is:

$$\Delta\xi_{31}^{new} = \Delta\xi_{31} + \frac{D_1}{\sin(a_{23} - a_{13})} \tag{4.17}$$

and

$$\Delta\xi_{32}^{new} = \Delta\xi_{32} + \frac{D_2}{\sin(a_{23} - a_{13})} \tag{4.18}$$

$$\Delta\eta_{31}^{new} = \Delta\eta_{31} + \frac{D_3}{\sin(a_{23} - a_{13})} \tag{4.19}$$

$$\Delta\eta_{32}^{new} = \Delta\eta_{32} + \frac{D_4}{\sin(a_{23} - a_{13})} \tag{4.20}$$

Since the second terms in the right hand side of (4.17) to (4.20), which also appear in (4.12) to (4.15), are not considered negligible in the classical solution, then they should not be considered negligible in the new solution.

5. Impact on the improvement of the deflections of the vertical – example

We chose three points in the broad Athens area, for which we have data from a gravity net established by the National Technical University of Athens. Point P_1 is located at the National Observatory of Athens, point P_2 at the Dept. of Topography of the National Technical University and point P_3 at the Dionysos Satellite Station. The relevant data are in Table 1:

Table 1. Coordinates and other data of the chosen points

Point	φ [°]	λ [°]	h [m]	g [mgal]	ξ [″]	η [″]
P_1	37.973210444	23.718125278	190.20	979950.543	−3.622	−10.093
P_2	37.975138889	23.780219444	244.00	979940.832	−2.826	−8.490
P_3	38.078595500	23.932465861	510.40	979778.625	−2.262	−0.025

The values (ξ, η) of the deflection of the vertical were derived from the EGM2008 gravitational model (*Hirt, 2010*) and were properly adjusted to the Earth's physical surface.

Other computed elements are as follows (Table 2):

Table 2. Distances, azimuths and deflection differences

$S_{12} = 5460.01$ m	$\alpha_{12} = 87.73362°$	$\Delta\xi_{12} = 0.796''$	$\Delta\eta_{12} = 1.603''$
$S_{23} = 17622.45$ m	$\alpha_{23} = 49.28850°$	$\Delta\xi_{23} = 0.564''$	$\Delta\eta_{23} = 8.465''$
$S_{13} = 22158.53$ m	$\alpha_{13} = 58.07052°$	$\Delta\xi_{13} = 1.360''$	$\Delta\eta_{13} = 10.068''$

The known parameters are the values of $\Delta\xi_{12}$ and $\Delta\eta_{12}$. Since the values of W_{xy} at the three points are not known, we can only compute the fractions of $\Delta\xi$ and $\Delta\eta$ from the first part of (4.12)–(4.15) (determinants D_1, D_2, D_3, and D_4) and compare them with the whole values (known from the model). In Table 3, which summarizes the results, we present the computed fractions of $\Delta\xi$ and $\Delta\eta$ from determinants D_1, D_2, D_3, and D_4, as explained above (denoted "D value").

Table 3. Results of the numerical example

	Model value [$''$]	Case I – D value [$''$]	Case II – D value [$''$]
$\Delta\xi_{13}$	1.360	−0.0011	−0.0022
$\Delta\xi_{23}$	0.564	0.0011	0.0022
$\Delta\eta_{13}$	10.068	−0.0015	−0.0030
$\Delta\eta_{23}$	8.465	0.0015	0.0030

From the results presented in Table 3 we see that the proposed refinement of the normal Eötvös matrix modifies the values of the deflections of the vertical by a few milliarcseconds.

6. Conclusions

In this paper we briefly presented the determination of the deflection of the vertical at points on the Earth's physical surface with the aid of torsion balance measurements. We used algebraic equations which include, amongst others, the second order partial derivatives of the actual gravity potential and the second order partial derivatives of the normal potential (the diagonal independent elements of the normal Eötvös matrix). The actual values are taken at points of interest on the Earth's physical surface, while the normal values are taken on the surface of the reference ellipsoid. We then outlined the method for the determination of the second order partial derivatives of the normal potential on the Earth's surface. In the last sections we produced analytical expressions for the computations of the elements of the deflection of the vertical. Finally, using a numerical example, we showed that the suggested refinement is small but not negligible.

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Appendix

Equation (3.17) can be further analysed to:

$$\begin{aligned}(W_{yy} - W_{xx}) - (U_{yy} - U_{xx}) &= (-gk_2^a + gk_1^a) - (\gamma k_2 - \gamma k_1) = \\ &= (\gamma^\alpha + \delta g)(k_1^a - k_2^a) - \gamma(k_1 - k_2)\,,\end{aligned} \tag{A.1}$$

where γ^α is the normal gravity value at point P_i and δg is the gravity disturbance at the same point. In addition:

$$\begin{aligned}(\gamma^\alpha + \delta g)(k_1^a - k_2^a) - \gamma(k_1 - k_2) = \\ &= \left(\gamma + \left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i}\right)(k_1^a - k_2^a) + \delta g(k_1^a - k_2^a) - \gamma(k_1 - k_2) = \\ &= \gamma(k_1^a - k_2^a) - \gamma(k_1 - k_2) + \left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i}(k_1^a - k_2^a) + \delta g(k_1^a - k_2^a)\,.\end{aligned} \tag{A.2}$$

The right hand side of equation (A.2) approximately becomes:

$$\begin{aligned}&\gamma\left(\frac{1}{R_1 + h_{P_i}} - \frac{1}{R_2 + h_{P_i}}\right) - \gamma\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \\ &\quad + \left(\left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i} + \delta g\right)\left(\frac{1}{R_1 + h_{P_i}} - \frac{1}{R_2 + h_{P_i}}\right),\end{aligned} \tag{A.3}$$

where R_1 and R_2 are the values of the principal radii of curvature of the ellipsoid at point Q_i. But:

$$\begin{aligned}&\frac{1}{R_1 + h_{P_i}} - \frac{1}{R_2 + h_{P_i}} = \frac{1}{R_1\left(1 + \frac{h_{P_i}}{R_1}\right)} - \frac{1}{R_2\left(1 + \frac{h_{P_i}}{R_2}\right)} \cong \\ &\quad \cong \frac{1}{R_1}\left(1 - \frac{h_{P_i}}{R_1}\right) - \frac{1}{R_2}\left(1 - \frac{h_{P_i}}{R_2}\right).\end{aligned} \tag{A.4}$$

Therefore, with the help of relation (A.4), relation (A.3) becomes:

$$\begin{aligned}&\gamma\left[\frac{1}{R_1}\left(1 - \frac{h_{P_i}}{R_1}\right) - \frac{1}{R_2}\left(1 - \frac{h_{P_i}}{R_2}\right)\right] - \gamma\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \\ &\quad + \left(\left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i} + \delta g\right)\left[\frac{1}{R_1}\left(1 - \frac{h_{P_i}}{R_1}\right) - \frac{1}{R_2}\left(1 - \frac{h_{P_i}}{R_2}\right)\right]\end{aligned} \tag{A.5}$$

After minor manipulations we have that:

$$\begin{aligned}&-\gamma h_{P_i}\left(\frac{1}{R_1^2} - \frac{1}{R_2^2}\right) + \left(\left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i} + \delta g\right)\left[\frac{1}{R_1} - \frac{1}{R_2} + \right. \\ &\quad \left. + h_{P_i}\left(-\frac{1}{R_1^2} + \frac{1}{R_2^2}\right)\right].\end{aligned} \tag{A.6}$$

For the principal curvatures it holds that (*Hofmann-Wellenhof and Moritz, 2006*):

$$\frac{1}{R_1} - \frac{1}{R_2} \cong -\frac{b}{a^2} e'^2 \cos^2 \phi \tag{A.7}$$

$$\frac{1}{R_1^2} - \frac{1}{R_2^2} \cong \frac{2b^2}{a^4} e'^2 \cos^2 \phi \tag{A.8}$$

Relation (A.6) becomes:

$$\begin{aligned} &-2\gamma h_{P_i} \frac{b^2}{a^4} e'^2 \cos^2 \phi - \left(\left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i} + \delta g(P_i) \right) \left[\frac{b}{a^2} e'^2 \cos^2 \phi + \right. \\ &\qquad \left. + h_{P_i} \frac{2b^2}{a^4} e'^2 \cos^2 \phi \,. \right. \end{aligned} \tag{A.9}$$

Rearranging the terms of the above relation we arrive at the following relation:

$$\begin{aligned} &(W_{yy} - W_{xx}) - (U_{yy} - U_{xx}) = \\ &\qquad = - \left(\gamma + \left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i} + \delta g(P_i) \right) \frac{2b^2}{a^4} h_{P_i}\, e'^2 \cos^2 \phi - \\ &\qquad\quad - \left(\left.\frac{\partial \gamma}{\partial z}\right|_{Q_i} h_{P_i} + \delta g(P_i) \right) \frac{b}{a^2}\, e'^2 \cos^2 \phi \end{aligned} \tag{A.10}$$

From relation (A.8) we have:

$$\varepsilon_1 = -\gamma (k_1^2 - k_2^2) h_{P_i} = \gamma \left(\frac{1}{R_2^2} - \frac{1}{R_1^2} \right) h_{P_i} \cong -2e'^2 \gamma \cos^2 \phi_{P_i} \frac{b^2}{a^4} h_{P_i} \,. \tag{A.11}$$

From (3.15), (A.10) and (A.11) (substituting the letter "z" with the letter "h" in the vertical gradient of normal gravity) we finally conclude that the improved term is equal to:

$$\begin{aligned} &(W_{yy} - W_{xx}) - (U_{yy} - U_{xx}) = \\ &\qquad = - \left(2\gamma + \delta g(P_i) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_i} h_{P_i} \right) \left(2e'^2 \frac{b^2}{a^4} \cos^2 \phi_{P_i} \right) h_{P_i} - \\ &\qquad\quad - \left(\delta g(P_i) + \left.\frac{\partial \gamma}{\partial h}\right|_{Q_i} h_{P_i} \right) \frac{b}{a^2}\, e'^2 \cos^2 \phi_{P_i} - \left.\frac{\partial^2 \gamma}{\partial y^2}\right|_{Q_i} h_{P_i} \,. \end{aligned} \tag{A.12}$$