On the existence of neutral directions of the normal gravity field

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Abstract: A neutral direction of a gravity field is a direction along which the components of the gravity vector remain locally unchanged. A neutral point is a point at which there exists a neutral direction. This research will focus on the neutral directions for the normal gravity vector. The necessary condition for the existence of neutral directions at an arbitrary point P above the ellipsoid is that the determinant of the Eötvös matrix must be equal to zero. The slopes of these directions depend on the value of the principal curvatures and the curvature of the plumbline. In all cases the neutral directions lie on the meridian plane at point P. An interesting case is when the vertical gradient of normal gravity is equal to zero. Finally in the last two paragraphs we show that neutral points are not isolated in the three dimensional space and give a numerical example for the case of a spherical gravity field.

Key words: neutral point, neutral direction, normal gravity field, normal gravity vector, Eötvös matrix

1. Introduction

Many interesting problems in physical sciences are related to 1) the investigation for invariant quantities of geometrical objects (for example the Gauss curvature) or 2) the investigation for quantities which remain unchanged on specific geometric objects. In the latter case some examples are a) equipotential surfaces (constant potential), b) isozenithal lines (gravity vector parallel to itself), and c) isocurvature lines (constant plumbline curvature). For isozenithal lines (related to Molodensky's problem) see for example *Moritz (1980)* and *Sanso (1978)*. *Moritz (op. cit.)* describes how isozenithal lines involve with the linearization of the boundary condition

of Molodensky's problem and gives their shape schematically for the case of the normal gravity field. *Sanso (op. cit.)* also describes the shape of isozenithal lines in a gravity field generated by a rotated sphere. An additional interesting work on isozenithal lines can be found in *Marussi (1985)* who defines two quantities (the curvature and torsion of the Earth's gravity field) and describes the connection between them and isozenithal lines. This connection results in an equation for the bearing of an isozenithal line. For isocurvature lines see *Manoussakis and Delikaraoglou (2011)*.

Another wide area in mathematics, which has affected geodesy, is the study of singularities. Significant work has been made on the existence of singularities in the Earth's gravity field as well as the Earth's normal gravity field. A famous problem in theoretical geodesy which involves algebraic singularities is the "geodetic singularity problem" (*Grafarend, 1971; Livieratos, 1976*). The geodetic singularity problem is related to the connection between the differentials of the astronomical coordinates (Φ, Λ, W) at a point P on the Earth's physical surface – where W stands for the Earth's gravity potential, Φ for astronomical latitude and Λ for the astronomical longitude – and the differentials of a local Cartesian system (x, y, z) such that the z–axis is vertical to the actual equipotential surface at point P, the x–axis points East and the y–axis points North. Both differential triads are related to a three dimensional matrix whose determinant depends on the magnitude of the actual gravity vector, the value of $\sec \Phi$, and the value of the Gauss curvature of the actual equipotential surface at point P. If this transformation matrix is rank deficient, i.e. its determinant is equal to zero, then it cannot be inverted. This deficiency results in the geodetic singularity problem.

Another example is the altimetry-gravimetry geodetic boundary value problem. The solvability of the linear fixed altimetry-gravimetry geodetic boundary value problem depends on the absolute value of the difference between the Gauss curvature of the ellipsoid of revolution and the Gauss curvature of the boundary surface which represents the Earth's surface (*Panou et al., 2013*). If this difference attains a specific value then it causes an analytical singularity which prevents the solvability of the problem. The Earth's normal gravity field (*Somigliana, 1929*) is extensively studied and used in Geodesy, so the singularity problem is also studied in the case for the Earth's normal gravity field (we will write only "normal gravity field" in

what follows). The symmetry properties of the normal gravity field make it useful in the study of various problems regarding the Earth's actual gravity field. Yet we have to mention that the normal gravity field is also very interesting by itself.

This work refers to the latter two cases. The quantities, which we are interested in, are the components of the normal gravity vector and under which conditions these remain unchanged.

2. Configuration of the problem

Let U be the normal potential of the gravity field generated by the level (normal) ellipsoid and P be an arbitrary point above the ellipsoid. Let UP be the value of the normal potential at this point. Suppose that (X, Y, Z) is a Cartesian rotating system such that the Z–axis is along the axis of rotation of the ellipsoid, the X–axis is the intersection of the zero meridian plane and the equator's plane and the Y-axis makes the system right-handed. On the normal equipotential surface passing through point P we define a second Cartesian system (x, y, z) such that its origin is at point P, the z–axis is normal to the equipotential surface, the y–axis points north and the x–axis points east. The vector equation of the normal equipotential surface for a region around point P has the form

$$\bar{s}(x, y) = (x, y, z(x, y)) \,, \tag{1}$$

where the value of the normal potential at P is not involved in the parameterization of the vector equation. The normal gravity vector (which represents the Newtonian and the centrifugal force) is vertical to the equipotential surface at P and its coordinates vary with the geodetic latitude and the geometrical height. But there are some cases in which the components of the gravity vector remain unchanged even if these two variables (latitude, height) are being changed. To be more specific, if some necessary conditions hold at point P, there is a direction on the meridian plane of that point along which the components of the normal gravity vector remain locally unchanged.

Let the Eötvös matrix at point P be

$$E_P = \begin{bmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{bmatrix}_P = \begin{bmatrix} -\gamma k_1 & 0 & 0 \\ 0 & -\gamma k_2 & -\gamma k \\ 0 & -\gamma k & 2\omega^2 + \gamma k_1 + \gamma k_2 \end{bmatrix}_P , \tag{2}$$

where the symbols ω, k_1, k_2 and k denote the angular velocity of the ellipsoid, the principal curvatures along the west-east direction, the north-south direction and the curvature of the plumbline respectively. If Q is an arbitrary point near point P and

$$\bar{\gamma}_P = (U_x, U_y, U_z)_P , \tag{3}$$

$$\bar{\gamma}_Q = (U_x, U_y, U_z)_Q , \tag{4}$$

$$\delta\bar{x} = (\delta x, \delta y, \delta z) = (x_Q, y_Q, z_Q) , \tag{5}$$

then

$$\begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}_Q = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix}_P + \begin{bmatrix} -\gamma k_1 & 0 & 0 \\ 0 & -\gamma k_2 & -\gamma k \\ 0 & -\gamma k & 2\omega^2 + \gamma k_1 + \gamma k_2 \end{bmatrix}_P \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} . \tag{6}$$

Since we are interested in the case that the components of the normal gravity vector remain unchanged at points P and Q it holds that (see also Eq. 5)

$$\begin{bmatrix} -\gamma k_1 & 0 & 0 \\ 0 & -\gamma k_2 & -\gamma k \\ 0 & -\gamma k & 2\omega^2 + \gamma k_1 + \gamma k_2 \end{bmatrix}_P \begin{bmatrix} x_Q \\ y_Q \\ z_Q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \tag{7}$$

The above system is a 3×3 algebraic system. The necessary condition for non-zero solutions is that its determinant is zero. The non-zero solutions are straight lines passing through point P which represent the neutral directions of the normal gravity vector. Hence the necessary condition in order to investigate the configuration of the neutral directions of the normal gravity vector is that the determinant of the Eötvös matrix at point P must be equal to zero. The detailed study is presented in the next section.

3. Existence of neutral points

The normal potential U is given by the formula (*Heiskanen and Moritz, 1967*)

$$U(u,\beta) = \frac{GM}{E}\tan^{-1}\frac{E}{u} + \frac{1}{2}\omega^2 a^2 \frac{q}{q_0}\left(\sin^2\beta - \frac{1}{3}\right) + \frac{1}{2}\omega^2(u^2 + E^2)\cos^2\beta\,, \quad (8)$$

where (u,β,λ) are the ellipsoidal coordinates which are given by the relations

$$X = \sqrt{u^2 + E^2}\,\cos\beta\,\cos\lambda\,,$$
$$Y = \sqrt{u^2 + E^2}\,\cos\beta\,\sin\lambda\,, \qquad\qquad (8a)$$
$$Z = u\,\sin\beta\,,$$

$$E^2 = a^2 - b^2\,. \qquad\qquad (8b)$$

The letters "a" and "b" stand for the semimajor and semiminor axis of the ellipsoid of revolution respectively, G stands for the gravitational constant and M is the Earth's mass. In addition,

$$q = q(u) = \frac{1}{2}\left[\left(1 + 3\frac{u^2}{E^2}\right)\arctan\left(\frac{E}{u}\right) - 3\frac{u}{E}\right]\,, \qquad\qquad (8c)$$

$$q_0 = q(b)\,. \qquad\qquad (8d)$$

The equipotential surfaces of the normal gravity field are closed convex surfaces. The normal potential is the sum of the Newtonian potential V and the centrifugal potential Φ. As we move further and further from the ellipsoid, the Newtonian potential is getting weaker and the centrifugal potential is getting stronger. Since the centrifugal force becomes stronger and it has the opposite direction from the Newtonian force eventually when we are far away from the ellipsoid we can find a point P such that

$$\gamma_P = |\mathrm{grad}\,U_P| = 0\,. \qquad\qquad (9)$$

The above relation means that at point P holds that

$$|\mathrm{grad}\,V_P| = |\mathrm{grad}\,\Phi_P|\,. \qquad\qquad (10)$$

At point P the magnitude of the normal gravity vector is equal to zero (we skip the letter P) i.e.

$$\gamma = 0\,. \qquad\qquad (11)$$

The above relation tells us that point P is an equilibrium point of the normal gravity field. Equilibrium points are neutral points of the normal gravity field. This can be seen easily since the determinant of the normal Eötvös matrix is equal to

$$-\gamma k_1 \left[-\gamma k_2 \left(2\omega^2 + \gamma k_1 + \gamma k_2 \right) - \gamma^2 k^2 \right] = 0 \ . \tag{12}$$

But generally it is not possible to investigate what kind of a neutral point may be, because the investigation depends on the kind of coordinate system we use. This will be clear in the following paragraphs. In addition, we will show that it is possible to find infinite many neutral points and we will show that they are not isolated in the three dimensional space. Supposing that a point P far away from the ellipsoid of revolution with non-zero normal gravity and such that

$$|\text{grad}\, V_P| < |\text{grad}\, \Phi_P| \ , \tag{13}$$

i.e. point P is further from the ellipsoid of revolution than an equilibrium point. Let U_P be the value of the normal potential of the equipotential surface passing through this point. Since the normal gravity vector has changed direction the Eötvös matrix at point P is now equal to

$$E_P = \begin{bmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{yx} & U_{yy} & U_{yz} \\ U_{zx} & U_{zy} & U_{zz} \end{bmatrix}_P = \begin{bmatrix} \gamma k_1 & 0 & 0 \\ 0 & \gamma k_2 & \gamma k \\ 0 & \gamma k & 2\omega^2 - \gamma k_1 - \gamma k_2 \end{bmatrix}_P . \tag{14}$$

The principal curvature along west-east direction at point P is also non-zero hence the determinant of the Eötvös matrix at point P becomes

$$\left[(k_1 + k_2)\, k_2 + k^2 \right] \gamma^2 - 2\omega^2 k_2 \gamma = 0 \ . \tag{15}$$

We will show that there exists a non-zero root for the above equation. Let ε be the vertical line to the ellipsoid passing through point P. Let Q be the intersection point of the vertical line ε and the ellipsoid. Supposing that (x_1, y_1, h) is a local Cartesian system whose origin is at point Q, the h–axis is normal to the ellipsoid of revolution, the y_1–axis points north and the x_1–axis points east. The point P is chosen to be very close to the equatorial plane therefore the local Cartesian systems at point P and Q are almost

parallel.

The Fig. 1 shows the two aforementioned local Cartesian systems at those points. The dotted line is the vertical line to the ellipsoid passing through point P. The determinant of the Eötvös matrix at point P can be also written as (we omit the letter "P")

$$U_{xx}\left(U_{yy}U_{hh} - U_{yh}^2\right) = 0 \ . \tag{15a}$$

Since the x second partial derivative of normal potential is non-zero, we expand the rest of the terms in Taylor series, using as a pole the point Q and up tp an odd integer m.

$$\left(U_{yy}(Q) + \sum_{n=1}^{m} \frac{1}{n!} \frac{\partial^n U_{yy}}{\partial h^n}\bigg|_Q h^n\right) \left(U_{hh}(Q) + \sum_{n=1}^{m} \frac{1}{n!} \frac{\partial^n U_{hh}}{\partial h^n}\bigg|_Q h^n\right) -$$
$$- \left(U_{yh}(Q) + \sum_{n=1}^{m} \frac{1}{n!} \frac{\partial^n U_{yh}}{\partial h^n}\bigg|_Q h^n\right)^2 = 0 \ . \tag{16}$$

After some manipulations, the above equation becomes an odd-degree algebraic equation of the form

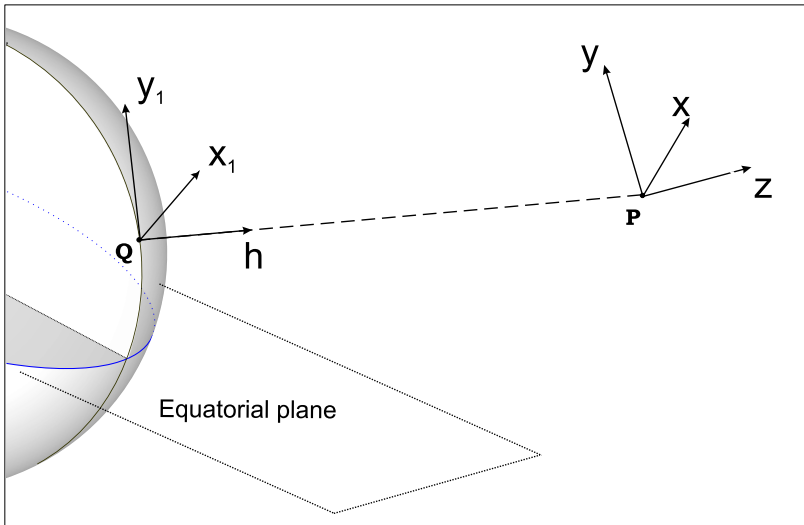Fig. 1. Local Cartesian systems at points P and Q.

$$\sum_{n=1}^{l} a_n h^n + a_0 = 0 \,, \tag{17}$$

where the coefficients a_0 and a_1, a_2,...,a_l are real numbers and l is an odd integer. The above equation has at least one real root which is the geometric height of the point P. Therefore point P is a neutral point. In the next section we will classify the neutral points according to the configuration of neutral directions and with the help of Bruns equation.

4. Configuration of neutral directions

Suppose that a point P is a neutral point above the equatorial plane and not an equilibrium point. As we mentioned in the previous paragraph, the determinant of the Eötvös matrix at point P is equal to zero. The Eötvös matrix is equal to

$$E_P = \begin{bmatrix} \gamma k_1 & 0 & 0 \\ 0 & \gamma k_2 & \gamma k \\ 0 & \gamma k & 2\omega^2 - \gamma k_1 - \gamma k_2 \end{bmatrix}_P . \tag{18}$$

The equation which can be used to determine the slope of the neutral direction on the meridian plane of point P is

$$k_2 y + k z = 0 \,. \tag{19}$$

This can be done because the determinant of the normal Eötvös matrix is zero, therefore the second and third rows are linearly dependent. Therefore a vector equation of the neutral direction on the meridian plane is

$$\varepsilon_n(y) = (0, y, z(y)) = \left(0, y, -\frac{k_2}{k} y\right) \,. \tag{20}$$

The Bruns equation at point P (*Heiskanen and Moritz, 1967*) is equal to

$$\left.\frac{\partial \gamma}{\partial z}\right|_P = -2\omega^2 + 2\gamma_P J_P \,, \tag{21}$$

where J_P is the mean curvature of the equipotential surface at this point. The Bruns equation is written with a changed sign because the neutral point P is further from the ellipsoid of revolution than the equilibrium point of the vertical line ε therefore the normal gravity vector has changed direction. At the neutral point P we have

$$\left| \left(\frac{\partial \gamma}{\partial z} \bigg|_P \right) \right| < 2\omega^2 \, . \tag{22}$$

As we mentioned before, along a neutral direction the coordinates of the normal gravity vector locally do not change. In this case point P is called *the neutral point of the first kind* (see Fig. 2).
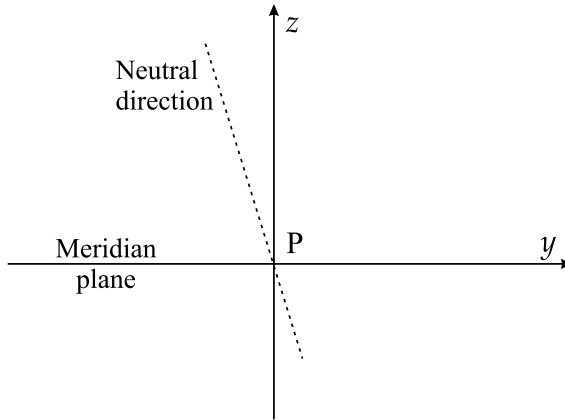

Fig. 2. Neutral direction at point P (γ_p not zero).

The second case is when point P is a neutral point on the equatorial plane. In this case the Eötvös matrix becomes

$$E_P = \begin{bmatrix} \gamma k_1 & 0 & 0 \\ 0 & \gamma k_2 & 0 \\ 0 & 0 & 2\omega^2 - \gamma k_1 - \gamma k_2 \end{bmatrix}_P \, . \tag{23}$$

After the equilibrium point of the equatorial plane the absolute value of the vertical gradient becomes smaller and smaller as we move further and further from the ellipsoid. Hence at point P on the equatorial plane it holds that

$$\frac{\partial \gamma}{\partial z}\bigg|_P = 0 \tag{24}$$

or

$$2\omega^2 - 2\gamma_P J_P = 0\,, \tag{25}$$

i.e. the determinant of the normal Eötvös matrix is equal to zero. A vector equation for the neutral direction at point P is

$$\varepsilon_n(z) = (0, 0, z)\,. \tag{26}$$

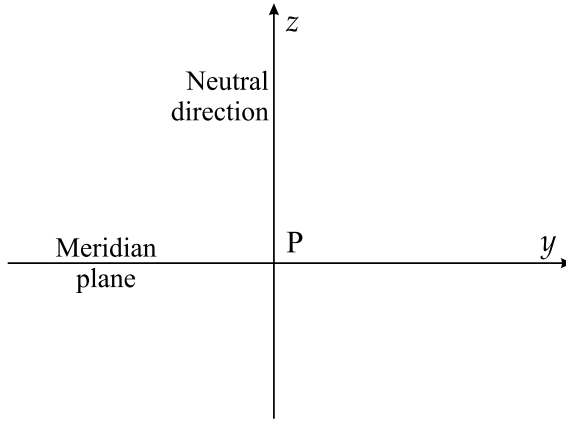In this case the point P is a *neutral point of the second kind* (see Fig. 3).

Fig. 3. Neutral direction at a point P on the equatorial plane (vertical gradient of normal gravity zero).

Last but not least is the case that point P is an equilibrium point. In this case the normal Eötvös matrix cannot be determined since (*Manoussakis, 2013*) the principal curvatures become infinite because the local Cartesian system which we use to study the neutral points is not the appropriate system for the study of equilibrium points of the normal gravity field.

5. Configuration of neutral points in space

We will continue our investigation by showing that if we find a neutral point then it is not isolated. Let P be a neutral point of the first kind which is

very close to the equatorial plane. Let Q be a point on the meridian plane of point P such that P and Q are very close. Since point P is a neutral point, the determinant of the Eötvös matrix is equal to zero. Since both points are very close the Eötvös matrix at point Q can be written as

$$E_Q = E_P + D_1\, y + D_2\, z\,, \tag{27}$$

where D_1 and D_2 are the disturbing matrices of the Eötvös matrix at point P. In a more analytical form relation (Eq. 27) is equal to

$$E_Q = \begin{bmatrix} U_{xx} & U_{xy} & U_{xz} \\ U_{xy} & U_{yy} & U_{yz} \\ U_{xz} & U_{yz} & U_{zz} \end{bmatrix}_P + \begin{bmatrix} U_{xxy} & U_{xyy} & U_{xzy} \\ U_{xyy} & U_{yyy} & U_{yyz} \\ U_{xzy} & U_{yyz} & U_{yzz} \end{bmatrix}_P y + \begin{bmatrix} U_{xxz} & U_{xyz} & U_{xzz} \\ U_{xyz} & U_{yyz} & U_{yzz} \\ U_{xzz} & U_{yzz} & U_{zzz} \end{bmatrix}_P z\,. \tag{28}$$

Substituting the necessary partial derivatives and changing some necessary signs (*Manoussakis, 2013*) the Eötvös matrix at point Q is equal to

$$E_Q = \begin{bmatrix} \gamma k_1 & 0 & 0 \\ 0 & \gamma k_2 & \gamma k \\ 0 & \gamma k & 2\omega^2 - \gamma k_1 - \gamma k_2 \end{bmatrix}_P +$$

$$+ \begin{bmatrix} \gamma\left(kk_1 + \dfrac{\partial k_2}{\partial y}\right) & 0 & 0 \\[2mm] 0 & \gamma\dfrac{\partial k_1}{\partial y} - 3\gamma k_2 k & -\gamma k_2^2 + \dfrac{\partial^2 \gamma}{\partial y^2} \\[2mm] 0 & -\gamma k_2^2 + \dfrac{\partial^2 \gamma}{\partial y^2} & -\gamma k(k_1 - 3k_2) - 2\gamma\dfrac{\partial J}{\partial y} \end{bmatrix}_P y \; +$$

$$+ \begin{bmatrix} \gamma k_1^2 & 0 & 0 \\[2mm] 0 & -\gamma k_2^2 + \dfrac{\partial^2 \gamma}{\partial y^2} & -\gamma k(k_1 - 3k_2) - 2\gamma\dfrac{\partial J}{\partial y} \\[2mm] 0 & -\gamma k(k_1 - 3k_2) - 2\gamma\dfrac{\partial J}{\partial y} & -\gamma(k_1^2 - k_2^2) - \dfrac{\partial^2 \gamma}{\partial y^2} \end{bmatrix}_P z\,. \tag{29}$$

The discriminant of the above matrix is

$$Det(E_Q) = \left[-\gamma k_1 + \gamma\left(kk_1 + \dfrac{\partial k_2}{\partial y}\right)y + \gamma k_1^2 z\right]\cdot \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix}\,, \tag{30}$$

where

$$D_{11} = \gamma k_2 + \left(\gamma \frac{\partial k_1}{\partial y} - 3\gamma k k_2\right) y - \left(\gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y^2}\right) z,$$

$$D_{12} = \gamma k - \left(\gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y^2}\right) y - \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right] z,$$

$$D_{21} = D_{12},$$

$$D_{22} = 2\omega^2 - \gamma k_1 - \gamma k_2 - \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right] y -$$

$$- \left[\frac{\partial^2 \gamma}{\partial y^2} + \gamma \left(k_1^2 - k_2^2\right)\right] z.$$

Expanding the above determinant we end up with the following second-order algebraic equation

$$\left\{\left[\frac{\partial^2 \gamma}{\partial y^2} + \gamma \left(k_1^2 - k_2^2\right)\right] \left(\gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y^2}\right) - \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right]^2\right\} z^2 -$$

$$- \left\{\left(\gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y^2}\right)^2 + \left(\gamma \frac{\partial k_1}{\partial y} - \gamma k k_2\right) \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right]\right\} y^2 -$$

$$- \left\{\left(\gamma \frac{\partial k_1}{\partial y} - \gamma k k_2\right) \left[\frac{\partial^2 \gamma}{\partial y^2} + \gamma \left(k_1^2 - k_2^2\right)\right] +\right.$$

$$\left. + \left(\gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y^2}\right) \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right]\right\} yz +$$

$$+ \left\{\left(2\omega^2 - \gamma k_1 - \gamma k_2\right) \left(\gamma \frac{\partial k_1}{\partial y} - \gamma k k_2\right) -\right.$$

$$\left. - \gamma k_2 \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right] + 2\gamma k \left(\gamma k_2^2 - \frac{\partial^2 \gamma}{\partial y^2}\right)\right\} y +$$

$$+ \left\{\left(2\omega^2 - \gamma k_1 - \gamma k_2\right) \left(-\gamma k_2^2 + \frac{\partial^2 \gamma}{\partial y^2}\right) -\right.$$

$$\left. - \gamma k_2 \left[\frac{\partial^2 \gamma}{\partial y^2} + \gamma \left(k_1^2 - k_2^2\right)\right] + 2\gamma k \left[\gamma k (k_1 - 3k_2) + 2\gamma \frac{\partial J}{\partial y}\right]\right\} z = 0.$$

(31)

The above equation does not contain the constant term $Det(E_P)$ because it is equal to zero. The above equation represents a conic which passes

through point P. For small values of the variable y the above equation is a segment of the conic which is close to point P. Along this segment all points are neutral points, i.e. points which have a neutral direction. Hence neutral points of the first kind are not isolated in the three dimensional space.

The segment of the conic which is close to point P is called *the local neutral curve of the first kind.* If we confine ourselves close enough to point P then we can omit the second order terms of the above equation. In this case we will form a linear equation which represents the tangent line of the local neutral curve at point P. This equation is equal to

$$
\begin{aligned}
&\left\{ \left(2\omega^2 - \gamma k_1 - \gamma k_2\right)\left(\gamma\frac{\partial k_1}{\partial y} - \gamma k k_2\right) - \gamma k_2 \left[\gamma k \left(k_1 - 3k_2\right) + 2\gamma\frac{\partial J}{\partial y}\right] + \right.\\
&\left. + 2\gamma k \left(\gamma k_2^2 - \frac{\partial^2\gamma}{\partial y^2}\right)\right\} y + \left\{ \left(2\omega^2 - \gamma k_1 - \gamma k_2\right)\left(-\gamma k_2^2 + \frac{\partial^2\gamma}{\partial y^2}\right) - \right.\\
&\left. - \gamma k_2 \left[\frac{\partial^2\gamma}{\partial y^2} - \gamma\left(k_1^2 - k_2^2\right)\right] - 2\gamma k \left[\gamma k \left(k_1 - 3k_2\right) + 2\gamma\frac{\partial J}{\partial y}\right]\right\} z = 0 \, .
\end{aligned}
\tag{32}
$$

Along the tangent line of point P we can choose another point, repeat the procedure and pinpoint another neutral point. If we do this many times, we will find a segmented line. This segmented line represents a curve on the meridian plane which is called *the neutral curve of the first kind.*

Now let point P be a neutral point on the equatorial plane at which the vertical gradient of normal gravity is equal to zero. Expanding the determinant given by equation (30) and (31) we obtain

$$
\begin{aligned}
&\left\{ \left[\frac{\partial^2\gamma}{\partial y^2} + \gamma\left(k_1^2 - k_2^2\right)\right]\left(\gamma k_2^2 - \frac{\partial^2\gamma}{\partial y^2}\right) - \left(2\gamma\frac{\partial J}{\partial y}\right)^2 \right\} z^2 - \\
&- \left\{ \left(\gamma k_2^2 - \frac{\partial^2\gamma}{\partial y^2}\right)^2 + 2\gamma^2 \frac{\partial k_1}{\partial y}\frac{\partial J}{\partial y} \right\} y^2 - \\
&- \left\{ \gamma\frac{\partial k_1}{\partial y}\left[\frac{\partial^2\gamma}{\partial y^2} + \gamma\left(k_1^2 - k_2^2\right)\right] + \left(\gamma k_2^2 - \frac{\partial^2\gamma}{\partial y^2}\right)2\gamma\frac{\partial J}{\partial y} \right\} yz - \\
&- \gamma k_2 2\gamma\frac{\partial J}{\partial y} y - \gamma k_2 \left[\frac{\partial^2\gamma}{\partial y^2} + \gamma\left(k_1^2 - k_2^2\right)\right] z = 0 \, .
\end{aligned}
\tag{33}
$$

The above equation represents the locus of neutral points locally around the neutral point P above the equatorial plane at which the vertical gradient

of normal gravity is equal to zero. If we confine ourselves in the vicinity of point P the equation of the conic becomes

$$\left(2\gamma^2 k_2 \frac{\partial J}{\partial y}\right) y + \gamma k_2 \left[\frac{\partial^2 \gamma}{\partial y^2} + \gamma \left(k_1^2 - k_2^2\right)\right] z = 0. \tag{34}$$

Finally we mention that it is not possible to study the equilibrium points in this manner but we know that they are isolated from the aforementioned neutral points.

In the Fig. 4 we show a section of the equatorial plane with a meridian plane. We point an equilibrium point P_e, a first kind neutral point P_1 and a second kind neutral point P_2. In addition, we draw the neutral curves schematically at each point. If the neutral curves are rotated around the ellipsoid then we will form the local shape of the neutral surfaces at these points. If we isolate the part of the neutral curve at point P_1 above the equatorial plane and rotate it around the ellipsoid then we will form the part of the neutral surface which is drawn in the next figure (see Fig. 5).
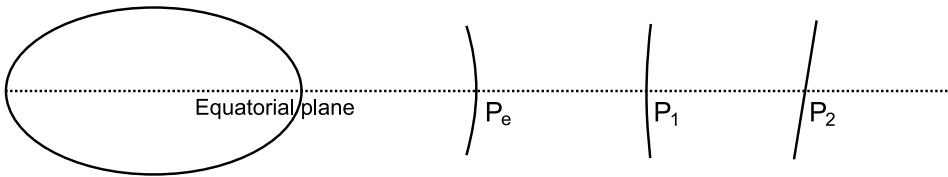

Fig. 4. Three possible cases of neutral curves.

Due to the symmetry of the normal gravity field the vector equation of the neutral surface in the system (X, Y, Z) is equal to

$$\bar{s}(y, \lambda) = \Big(X(y, z(y)) \cos \lambda,\ Y(y, z(y)) \sin \lambda,\ Z(y, z(y))\Big). \tag{35}$$

An equilibrium point is a neutral point and it is isolated from the other neutral points. Due to the complexity of the normal gravity field, we will determine the equilibrium surface of a spherical gravity field in the next section.

Figure 5 shows the local shape of a neutral surface of the first kind generated from a neutral curve at the first kind neutral point P. At the center of the above figure is the ellipsoid of revolution. In the case of a

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neutral point of a second kind the rotational neutral surface is a segment of a cone.

We summarize the following results:

a) Neutral points of the first kind are not isolated in the three dimensional space and they form a curve on the meridian plane which is the neutral curve of the first kind. This curve is the generator curve of a rotational surface which is called neutral surface of the first kind.

b) Neutral points of the second kind are also not isolated from each other and they are isolated from the neutral points of the first kind. Neutral points of the second kind form a rotational surface which is called a neutral cone.

c) Equilibrium points are also neutral points but in the case of the normal gravity field it is not possible to investigate them in the chosen coordinate system. Equilibrium points are not isolated from each other but they are isolated from the neutral points of the first and second kind.
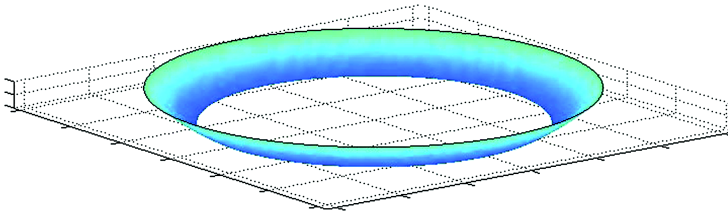

Fig. 5. Neutral surface (hyperbolic neutral point).

6. Numerical example: Neutral points of a spherical gravity field

In this section we determine the neutral points of a spherical gravity field. Let S be a sphere which approximates the figure of the Earth with $R = 6371$ km, contains the mass of the Earth and rotates with the Earth's angular velocity. Supposing a Cartesian rotating system (X, Y, Z) such that Z–axis is the Earth's mean axis of rotation, the X–axis is the intersection of the meridian plane of Greenwich and the equator's plane and the Y–axis makes the system right-handed. If P is a point on the equatorial plane of the sphere then its total gravity potential is equal to

$$U_S(P) = V_P + \Phi_P = \frac{GM}{r_P} + \frac{1}{2}\omega^2 r_P^2 , \qquad (36)$$

where r_P is the distance of point P from the center of the sphere S. According to section 2 equilibrium points are neutral points. On the equatorial plane we form the following equation

$$\frac{GM}{r_P^2} = \omega^2 r_P . \qquad (37)$$

The only unknown is the variable r_P, therefore the distance of an equilibrium point (which is also a neutral point) on the equatorial plane from the center of the sphere is equal to

$$r_P = \sqrt[3]{\frac{GM}{\omega^2}} = 42164.173 \text{ km} . \qquad (38)$$

The equipotential surfaces of the spherical gravity field are spheres and at each point the principal curvatures along west-east and north-south direction are equal. Supposing a local Cartesian system (x, y, z) whose origin is a point P on the meridian, the z–axis is vertical to the equipotential surface at point P, y–axis is tangent to the equipotential surface at point P and points north, and x–axis is tangent to the equipotential surface and points east. Since the plumblines of the spherical gravity field are straight lines the Eötvös matrix at point P is equal to

$$E_P = \begin{bmatrix} \gamma_S k_S & 0 & 0 \\ 0 & \gamma_S k_S & 0 \\ 0 & 0 & 2\omega^2 - 2\gamma_S k_S \end{bmatrix}_P , \qquad (39)$$

where k_S is the curvature of the spherical equipotential surface. The Eötvös matrix is expressed in the local Cartesian system; therefore it cannot be determined at an equilibrium point.

Now we investigate if there is a neutral point of the second kind on the equatorial plane. In order to find it, we have to solve the following equation

$$2\omega^2 - 2\gamma_S k_S = 0 \qquad (40)$$

and since $k_s = 1/r$, then

$$\gamma_S = \omega^2 r \,. \tag{41}$$

The above equation means that the Newtonian part of gravity must be equal to zero. Of course this is not possible so that in the spherical gravity field we do not have neutral points of the second kind.

Finally we are going to find the equation of the equilibrium surface. Generally at a equilibrium point above the equatorial plane it holds that

$$\frac{GM}{(X^2 + Y^2 + Z^2)} = \omega^2 \sqrt{X^2 + Y^2 + Z^2} \cos\phi \,, \tag{42}$$

where ϕ is the spherical latitude. But

$$\cos\phi = \frac{\sqrt{X^2 + Y^2}}{\sqrt{X^2 + Y^2 + Z^2}} \,. \tag{43}$$

Making the necessary manipulations and confining ourselves to the first quadrant we obtain

$$Z = \sqrt{\frac{GM}{\omega^2 \sqrt{X^2 + Y^2}} - (X^2 + Y^2)} \,. \tag{44}$$

Making the transformation

$$r_e = \sqrt{X^2 + Y^2} \,, \tag{45}$$

the parametric equation for the equilibrium curve in the first quadrant is

$$\bar{r}_S(r_e) = \left(r_e, \sqrt{\frac{GM}{\omega^2 r_e} - r_e^2} \right) , \qquad r_e \in (0, \ 42164.173 \ \text{km}) \,, \tag{46}$$

and a vector equation for the part of the equilibrium surface for the first quadrant is

$$\bar{s}(r_e, \lambda) = \left(r_e \cos\lambda, r_e \sin\lambda, \sqrt{\frac{GM}{\omega^2 r_e} - r_e^2} \right) . \tag{47}$$

Figure 6 is quite informative for the shape of the equilibrium surface in the three dimensional space. It shows an intersection of the equilibrium surface
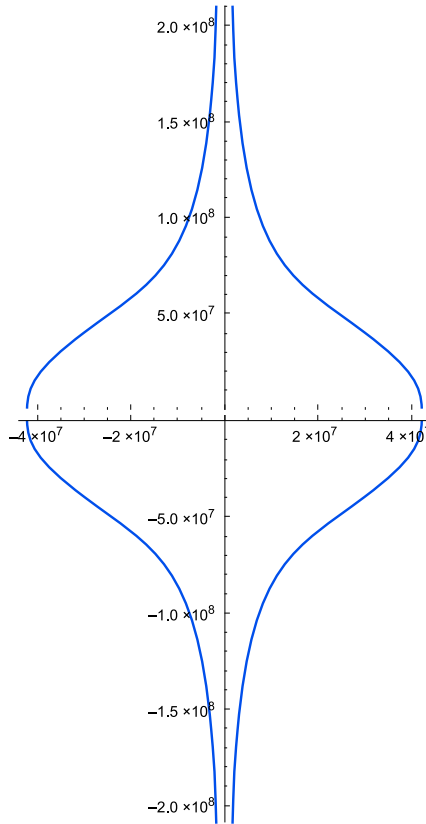
Fig. 6. Section of the equilibrium surface with a meridian plane.

with a meridian plane. The equilibrium surface is a rotational surface and its points are equilibrium points hence neutral points. The units used are meters.

7. Conclusions

In this work, we have investigated the existence of neutral directions, i.e. the existence of neutral points of the normal gravity vector and we described their configuration in each case. The neutral points were classified into

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two categories. It is quite clear that the Eötvös matrix is very significant since it contains valuable information for the local behavior of the normal gravity vector. That is to say, that a neutral direction can be found only when the determinant of the normal Eötvös matrix at a point P is equal to zero. Interesting cases are when at a neutral point the vertical gradient of normal gravity is equal to zero or it is an equilibrium point. We have also showed that neutral points of the same kind were not isolated points in the three dimensional space but different kind of neutral points were isolated. In the case of equilibrium points we cannot investigate what kind of neutral points are because of the choice of the coordinate system. Neutral points lie on special plane curves called neutral curves. The rotational symmetry of the normal gravity field results in the existence of special rotational surfaces which are called neutral surfaces. In the case of equilibrium points the surface is called equilibrium surface. Due to the complexity of the normal gravity field we presented a numerical example which involved the determination of neutral points of a spherical gravity field. The spherical gravity field is generated by a sphere which contains the Earth's mass and its radius is the mean Earth radius then there exists only an equilibrium surface whose points are simultaneously neutral points. Since the ellipsoid of revolution has very small flattening we can make the assumption that the equilibrium points of the normal gravity field on the equatorial plane will have a similar distance from the center of the ellipsoid with those of the spherical gravity field. Finally we form a vector equation of the equilibrium surface for the case of a spherical gravity field and made an informative figure. Neutral directions represent an additional interesting property of the normal gravity field which promotes the significance of the Eötvös matrix.

References

Grafarend E., 1971: The object of anholonomy and a generalized Riemannian geometry for Geodesy. Boll., Geof., Teor., Applic., 13, 241–253.

Heiskanen W. A., Moritz H., 1967: Physical Geodesy. W. H. Freeman, San Francisco, p. 53.

Livieratos E., 1976: On the geodetic singularity problem. Manuscripta Geodetica, I, 269–292.

Moritz H., 1980: Advanced Physical Geodesy. Herbert Wichmann Verlag.

Manoussakis G., Delikaraoglou D., 2011: On the gradient of curvature of the plumblines of the Earth's normal gravity field and its isocurvature lines. Studia Geophysica et Geodetica, **55**, 3, Springer Netherlands, 501–514.

Manoussakis G., 2013: Estimation of the normal Eötvös matrix for low geometric heights. Acta Geodaetica et Geophysica, **48**, 2, Springer, 179–189.

Marussi A., 1985: Intrinsic Geodesy. Translated by W. I. Reilly, Springer Verlag, 87–91.

Panou G., Yannakakis N., Delikaraoglou D., 2013: An analysis of the linear fixed altimetry-gravimetry boundary-value problem. Studia Geophysica et Geodetica, **57**, 2, 203–216.

Sanso F., 1978: Molodensky's problem in gravity space, a review of first results. Journal of Geodesy, **52**, 1, 59–70.

Somigliana E., 1929: Teoria generale del campo gravitazionale dell' ellipsoide di rotazione, Mem. Soc. Astron. Italia, IV, Milano.